KINEMATICS OF MACHINERY.

A BRIEF TREATISE ON CONSTRAINED MOTIONS OF MACHINE ELEMENTS.

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PREFACE TO THE SECOND EDITION

In revising this book several entirely new articles have been added to the original text, notably those on helical gears, and methods of gear cutting. A number of articles have been either wholly or partly rewritten, especially those dealing with instant centers, velocity and acceleration diagrams, rolling hyperboloids, involute teeth, and epicyclic trains. Many minor errors have been corrected, and several figures have been redrawn.

Professor Leslie D. Hayes has given valuable assistance, both in reading the proof and in the work of revision.

E. H. Wood.

Ithaca, New York,
January, 1911.

PREFACE TO THE FIRST EDITION.

THIS book is the outgrowth of a somewhat smaller treatise which was prepared and printed by the writer in 1894 for the use of the classes in mechanical and electrical engineering at Sibley College, Cornell University.

After having used the original for several years, it was decided to issue the work in revised form, making such corrections and changes as experience suggested.

The present volume was prepared especially to bring together, and to present to the students in a condensed text-book, those principles and methods which are deemed most important in a general course on Kinematics. This is the only excuse offered for another book on a subject about which so much has been written. No pretension is made to originality except in the arrangement and manner of presenting a few subjects. Neither is the present work offered as in any sense a complete treatise on the Kinematics of Machinery. The treatment of many topics has been much abridged; particularly the portion relating to toothed gearing, a subject which is exhaustively treated in numerous available works. On the other hand, the discussions of the applications of such important conceptions as instantaneous centres, velocity diagrams, etc., are rather fuller than are found in many of the shorter works on Mechanism.

The treatment of these subjects follows closely that given by Professor Kennedy in his admirable work on the Mechanics of Machinery.

It is believed that the presentation of principles and methods, with illustrations of their applications, is the proper line to adopt

in a text-book intended for a short general course on such a subject as Kinematics. The detailed description of usual forms, and the discussion of the innumerable considerations with which the expert in any line must be familiar are to be sought in special treatises.

Messrs. A. T. Bruegel, D. S. Kimball, and W. N. Barnard, all of whom have given instruction in the course to which it applies, have rendered valuable assistance in the preparation of the present book. Mr. Bruegel contributed most of the problems, which were developed during his six years as instructor in Kinematics at Cornell University. Professor Kimball kindly wrote the articles on "Acceleration Diagrams" and "Epicyclic Trains," and he and Mr. Barnard have cooperated in other ways in the revision.

Many earlier works have been consulted and drawn on in the preparation of the present book. The following, especially, should be mentioned: Principles of Mechanism, by Professor Willis; Machinery and Millwork, by Professor Rankine; Kinematics of Machinery, by Professor Reuleaux; Mechanics of Machinery, by Professor Kennedy; Kinematics, by Professor MacCord; Machine Design, by Professor Unwin; Elementary Mechanism, by Professors Stahl and Woods; Teeth of Gears, by Mr. George B. Grant; A Practical Treatise on Gearing (Beale), published by the Brown and Sharpe Manufacturing Company.

The writer desires to acknowledge his obligations to all who have in any way aided in the preparation of this little book.

JOHN H. BARR.

ITHACA, NEW YORK,
October 1899.

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KINEMATICS OF MACHINERY.

CHAPTER I.

FUNDAMENTAL CONCEPTIONS OF MOTION. THE NATURE OF A MACHINE.

1. **Motion** is a change of position; and it is measured by the space traversed. Time is not involved in this conception. A train, in running between two stations fifty miles apart, has the same motion, whether the time occupied be one, two, or three hours. The motion of a crank-pin in making a revolution is independent of the time required.

2. **Linear Velocity**, or simply velocity, is the rate of motion of a point along its path in space. It is a function of both space and time, and is measured in compound units of these fundamental quantities; as feet per second, feet per minute, miles per hour, etc. In mathematical terms, velocity $= v = \frac{ds}{dt}$, in which $s =$ the space passed over in the time t.

If, in the illustration of the preceding article, the time of the run between the stations is one hour, the train has an average, or mean, linear velocity of fifty miles per hour; if the time be two and a half hours, the mean velocity, or speed, as it is often called, is twenty miles per hour, etc., or 1760 ft. per min., or 29′ 4″ per sec.

3. **Acceleration**, or linear acceleration, is the *rate* of change of *velocity*. Acceleration is expressed in the same system of space- and time-units as the velocity itself (as feet and seconds, feet and min-

utes, miles and hours, etc.); but acceleration involves one space-factor and *two* time-factors. The mathematical expression for acceleration is $p = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

If a velocity is uniformly increased from 10 feet per second to 18 feet per second, the change of velocity is 8 feet per second. If this change takes place in 2 seconds, the rate of change, or the acceleration, is 4 feet per second per second, or 4 foot-seconds per second, or 4 feet per square second. If the increase of velocity is not uniform, the *mean* acceleration is 4 feet per square second in the above illustration, although the actual increase of velocity in any one second is not necessarily 4 feet per second.

4. Uniform and Variable Velocity.—If the motion of a body is uniform (that is, if all equal increments of space are traversed in equal increments of time) the velocity is uniform, and is equal to the space traversed in any time divided by that time. If the velocity is uniform, the acceleration is zero. If a body moves 120 feet in 10 seconds, with a uniform velocity, the velocity is 12 feet per second, equivalent to 720 feet per minute.

If the velocity is not uniform, the space divided by the time gives only the *mean* or average velocity, and the velocity may vary between the widest limits during the motion. If the law of the motion is known, the velocity at any instant may be determined from the space and time; otherwise, only the mean velocity can be determined from these data.

The velocity of a body may vary uniformly, the velocity increasing or decreasing by equal increments with each equal increment of time, in which case the acceleration is constant; or it may vary according to any other law. For our present purposes it is only necessary to discriminate between uniform, or constant, and varying velocity.

Although the velocity may be constantly changing, it is customary to speak of a body as moving at a certain velocity, as 25 feet per second, 30 miles per hour, etc.; and such expressions are perfectly correct, even though the velocity does not remain constant for a single instant. For example: a train of cars in getting up speed passes through every velocity from zero to the maximum

velocity attained; at a certain stage the velocity may be, say, ten miles per hour, and in coming to rest the velocity again passes through this same value. Perhaps the train does not maintain this particular velocity for a single foot; yet, for the instant, it is said to have this velocity; meaning that if it continued to move with the velocity that it has at this instant it would move 10 miles in one hour.

5. Relative and Absolute Motion.—All known motions are *relative*, for change of position can only be noted with reference to objects at rest (or assumed to be at rest), or by reference to objects the motion of which is known (or assumed to be known). We know of no body absolutely at rest, nor do we even know the absolute motion of any body in the universe.

In treating of the motion of a body, only its change of position with regard to some other body, or its *motion relative to that other body*, can be considered.

In ordinary problems of terrestrial mechanics the earth is taken as the standard from which to reckon, and a body which does not change its position relative to the earth is said to be at rest, stationary, or fixed ; of course recognizing that it partakes of the motion which the earth has about its axis, around the sun, and in common with the sun through space.

In problems of machinery the motions of the parts are usually most conveniently taken with reference to the frame of the machine as a standard. In "stationary" or "fixed" machines this is equivalent to referring these motions to the earth, for the frame has no appreciable motion relative to the earth; but in such cases as locomotives and marine engines, for example, the parts have very different motions relative to the frame and to the earth. In these latter cases we are usually concerned with the motion of the *parts relative to the frame*, or with the motion of the *machine as a whole* (including everything connected with it) *relative to the earth.*

The function of the machine, in these cases, is to impart motion, relative to the earth, to the attached train or ship, and incidentally to itself; but this motion of the entire system, and the motion of the parts, as members of a machine, may generally be treated as quite distinct, though related, problems. A marine engine can be

studied as an engine just as a mill engine can be treated, without considering the application of the energy beyond the engine itself.

As we know nothing of the absolute motion of a body, and can only know its motion relative to other objects, it can have as many relative motions as there are objects with which to compare its changes of position.

A pair of locomotive drivers, for example, rotate on their axis relative to the frame; they roll along the rails (each point tracing a curve of the cycloidal class) relative to the rails or the earth; they rotate about the axes of their pins relative to the attached side rods; and have still different motions relative to the wheels on the other axles, to the piston, etc.

It is important to get a clear conception of relative motion, for in the study of mechanism the treatment may often be much simplified by referring a motion to some member other than the frame of the machine, as to other moving parts.

Throughout this work it will frequently happen that the motion of a part relative to some other moving part will be discussed; but it is to be understood, unless distinctly indicated to the contrary, that the word "motion" refers to the change of position relative to the frame. Likewise, when a member is said to be at rest, fixed or stationary, it is to be understood that its position relative to the frame remains unchanged.

Two portions of a rigid body can have no motion relative to each other; for a change of the relative positions of such parts involves a change of form, and this is not consistent with the conception of a rigid body. It will be evident, upon brief reflection, that two separate bodies which have no relative motion could be rigidly joined without affecting any motions that they may have; for as they do not change their position relative to each other they must have identical motions relative to all other bodies, and may be treated as parts of the same body so far as their motions are concerned.

Bodies which have no motion relative to each other have the same motion relative to any other body.

The converse of this statement, that all bodies which have the same motion relative to another body have no motion relative to

each other, is not, however, generally true. Take the example of the locomotive driving-wheels, again; each set of wheels has the same motion relative to the frame, as well as to the rails, but the wheels on the different axles do, nevertheless, have motions relative to each other; for these different sets of wheels could not be rigidly fastened together as one piece without preventing motion relative to the frame.

6. Velocity Ratio.—In many problems of machine motions, the actual velocity of the parts is not of so much importance as the *ratio* of the velocities of two or more parts. In another class of problems, the actual velocity (relative to the earth, or other standard) must be treated. The present work is concerned very largely with the former class, and it is necessary to get a clear conceptiou of the term velocity ratio. This may perhaps be best accomplished by a few illustrations.

Take, as an example, an ordinary simple hand-windlass, in which a rope is wrapped around a drum of known size, and a crank of given radius is attached to the axis of the drum. If the crank be turned through one complete revolution, the load attached to the rope will be raised a height equal to the circumference of one coil. For any number of turns of the crank, or fractional turns, the load will be raised a proportional height; and it matters not whether the crank be turned fast or slowly, the *ratio* of its motion, and of its velocity, to that of the load is the same, depending entirely upon the proportions of the device. The ratio of the velocities, or the *velocity ratio*, is independent of the actual velocities, and of the forces transmitted. In the case cited, the ratio is the same whether a load of one ton be hoisted ten feet in one second, or one pound be hoisted one foot in one minute. The same point is illustrated in the action of most of the common machines. In an ordinary steam-engine, for every revolution of the crank the connected parts go through certain definite motions; while the time of one such revolution of the crank may be a tenth of a second or ten minutes, all the parts (with the exception of such parts as the members of the governor, to be mentioned later) go through the same relative changes of position; and though the actual velocities with which

such changes take place are very different in the two cases, the ratio of these velocities remains the same.

7. Path.—A point in changing its position traces a line called its path.

The statements in the preceding articles on the motions and velocities of *bodies* apply equally to every point in a moving body, whether the path of the point be rectilinear or otherwise. This is consistent with the definitions of motion and velocity; for these definitions state that motion is measured by space traversed (not restricted to space in a right line), and that velocity is the rate of motion.

The path of a point may be of any form whatever, in a plane or in space; it may be a straight or curved line of finite length, along which the point moves from end to end, reversing its direction of motion at either end, so that it passes any particular position first in one direction and then in the opposite direction; it may be a closed curve so that, unless the curve crosses itself, successive passings of any position are always in the same direction; or it may be an infinite straight or curved line, the point never twice occupying the same position, except in the special case in which the curved path crosses itself. There are many cases, however, in which the path is definite and limited in both form and extent, and nearly all motions of mechanisms are of this class.

8. Cycle; Period; Phase.—In most mechanisms the members go through a series of relative motions, at the end of which they occupy the same relative positions as at the beginning.

The completion of such a series of relative motions, with the return of the members to the relative positions which they had at first, constitutes a *cycle*.

In the ordinary steam-engine, for example, the cycle corresponds to one revolution of the crank, whatever the time occupied by the revolution. In a common type of gas-engine, the cycle corresponds to two complete revolutions of the crank, for the four strokes of the piston during these two revolution are: a suction stroke; a compression stroke; a working stroke (impulse); and an exhaust stroke. The valve-gear, in this case, is so arranged that valve, piston, etc.,

only return to their initial relative positions after the completion of four strokes of the piston, or two revolutions of the crank.

The time elapsing during a cycle is called the *period.*

The simultaneous positions occupied by the members, at any instant during the cycle, constitute a *phase.*

9. Continuous, Reciprocating, and Intermittent Motion.—If the direction of motion does not reverse, the motion is sometimes said to be *continuous* (using the word somewhat differently ~~than in~~ from the strict mathematical sense, in which all motion is continuous).

Motion is said to be *reciprocating* if its direction reverses.

Motion is called *intermittent* when it is interrupted by intervals of rest.

Motion in a closed path may be continuous, reciprocating, or intermittent; and it may vary as to velocity in any manner whatsoever.

Motion in a path of finite extent, not forming a closed figure, must be reciprocating, and may or may not be intermittent.

10. Plane Motion; Rotation, Translation.—Of the great number of motions available in machinery, a very large proportion are included in three classes of comparatively simple nature, viz.: Plane Motion, Helical Motion, and Spherical Motion.

Plane Motion is by far the most common, and it is the simplest class as well.

If any plane section of a body moves in its own plane, all points in this section move in this plane, and all points outside of this section move in planes parallel to the given section. Such a motion constitutes a plane motion. Any point in a body having plane motion may trace any path in its plane; but all points similarly located in the other parallel planes, that is all points lying in a common perpendicular to the different planes of motion, have paths of identically the same form. Thus, in Figs. 1 or 2, if the section shown shaded always moves in its own plane, the successive positions of the perpendicular through any point as p must always be parallel, and therefore all points in this perpendicular move in equal paths.

The property of plane motions, just discussed, greatly simplifies the treatment of these motions, as the motion of one point (or of a set of points) in any section represents the motion of all similar

points in other sections; or the motion of a single section (a plane figure) in its own plane represents the motion of the entire body. This can be extended even farther, for the motion of a point not in the particular plane represented can be replaced by that of its corresponding point on that plane (its projection on the plane), and thus the motion of a single plane figure represents all the

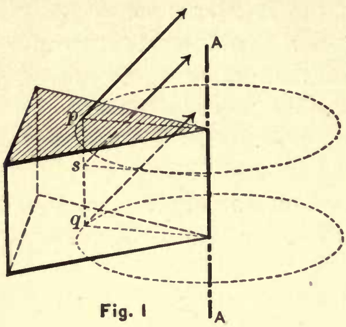

Fig. 1

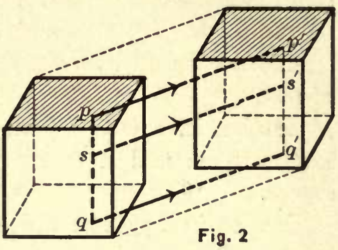

Fig. 2

motions of all the points in the body. For example, the motions of the points *p*, *s*, and *q* in Figs. 1 or 2, are in equal paths, and the motion of any one of these points may be taken to represent that of any other. The motion of an engine crank and of the eccentric can be, and often are, conveniently shown together, as if actually in one plane.

In case of all other than plane motions, however, it is necessary to show the various positions of the members by two or more projections, or by some equivalent system, if it is desired to completely represent the motion.

Plane Motion is either a *Rotation*, a *Translation*, or a motion which can be reduced to a *combination of these*. The reduction of the general motion to a combination of rotation and translation is not always to be desired, however, and such motion will often be treated as a class of itself, without relation to the simpler and more special classes to which it can be reduced.

If a body moves, as in Fig. 1, so that all points travel in parallel planes and at constant distances from a fixed right line, it has a *plane* motion of *rotation*. Examples: pulleys, cranks, levers, etc. It is not necessary that the motion be continuous; the rotation may be continuous, reciprocating, or intermittent.

If a body moves, as in Fig. 2, so that all points move with equal velocities in equal paths, the motion is a *translation*. If these paths are parallel right lines, the motion is a *rectilinear translation*. Examples: the carriage of a lathe, piston or cross-head of an engine, platen of a planer, etc. If, however, the paths of all the different points are equal curves, the motion is a *curvilinear translation*. Example: the side rods of a locomotive.

Rectilinear translation is always to be understood when the word translation is used without qualification.

A rectilinear translation may be treated as the special case of rotation in which the distance to the axis is infinity, or as rotation in a circle of infinite radius.

It has been shown that the plane motion of a body is completely represented by the motion of any section taken in a plane of motion, or by the change of position of a plane figure. Two points suffice to locate a figure in a plane, and hence the plane motion of a body is determined by the motion (successive positions) of any two of its points not in the same perpendicular to the plane of motion. In the general case of motion in space, the motion is determined only by the motions of at least three points, not in one right line. For if the motions in space of two points are known, the body may, in the general case, have a motion of rotation about the line connecting these two points; but the motion of a third point, outside of this line, determines the motion of the body completely.

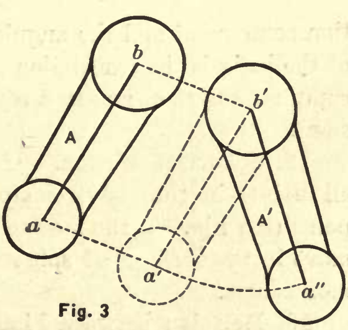

Fig. 3

In general, the motion of a body in a plane may be reduced to an equivalent rotation and a translation. Thus, Fig. 3, the motion of the body *A*, which is completely determined by the motion of two points such as *a* and *b*, or by the motion of the line connecting these points, corresponds to a change of position from *A* to *A′*. This change of position can be conceived as made up of a translation, *a-b* to *a′-b′*, and a rotation, about *b′* through the angle

$a'\ b'\ a''$. Or, the rotation can be conceived to take place first, followed by the translation.

As this motion is a perfectly general case of plane motion, the same reasoning applies to all such cases, no matter how large or how small the motion may be.

11. Helical Motion.—If all the points in a body have a motion of rotation about an axis, combined with a translation parallel to that axis, the motion is a *Helical Motion* (see Fig. 4). In nearly all cases the helical motions met with in machines are *regular helical motions*, in which there is a constant relation between the rotation and the translation; that is, the ratio between the transla-

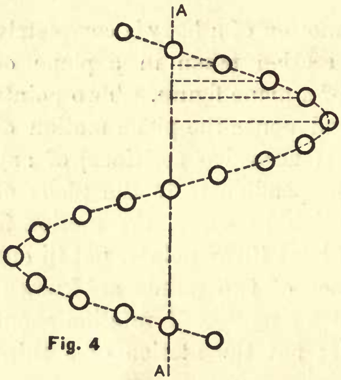

Fig. 4

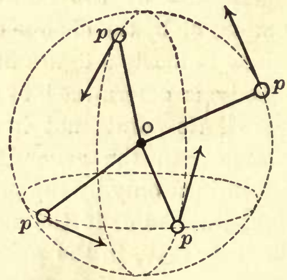

Fig. 5

tion component and the angular component is constant. The *pitch* of the helix is the translation along the axis corresponding to one complete rotation, and in a regular helical motion the pitch is constant.

12. Spherical Motion.—If the motion of a body is such that all points in the body remain at constant distances from a fixed point (see Fig. 5), the motion is spherical. All points in the body move in the surfaces of spheres, having the fixed point for a common centre.

13. Relation between Plane, Helical, and Spherical Motions.—If the translation component (*pitch*) in a helical motion be increased till it equals infinity, the motion reduces to a plane translation. On the other hand, if the translation component be re-

duced to zero, the motion reduces to plane rotation. It is thus seen that both of the limits of helical motion are plane motions, and that plane motion of rotation or of translation may be treated as special cases of helical motion.

If the distance from the fixed point to the moving body in a spherical motion be increased to infinity, the surfaces of the spheres in which the points of the body move are reduced to planes, and we thus see that plane motion may be treated as a special case of spherical motion. The much greater frequency of plane motion and its simplicity makes its consideration, in practical cases, as a

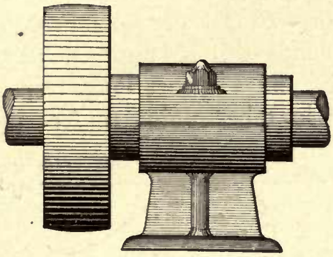

Fig. 6

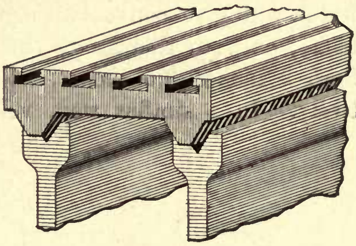

Fig. 7

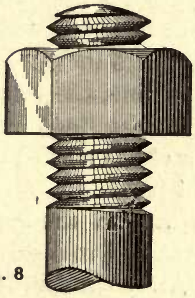

Fig. 8

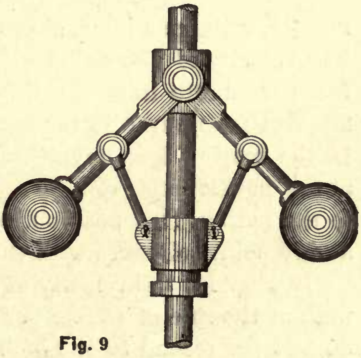

Fig. 9

special form of these more complex motions undesirable, though this view of the case is not without interest.

Motions more complicated than the classes just mentioned are sometimes met with in machinery, and some of these will be discussed in subsequent articles; but they are comparatively so

few, and are so varied in character, that a classification of them is not practicable. Figs. 6, 7, 8, and 9 show practical examples of plane rotation, plane translation, helical motion (regular) and spherical motion respectively.

14. Graphic Representation of Velocity.—The direction and velocity of a motion may be represented by a right line, the direction of which indicates the direction of the motion, while its length represents the velocity to some convenient scale.

If, for instance, it is desired to represent the velocities: 20 feet per second, 35 feet per second, 55 feet per second, and 40 feet per second, by lines on a drawing, or diagram, a scale can be adopted which will give convenient lengths (say 10 feet per second to the inch); and, to this scale, these velocities will be represented by lines 2 inches, 3.5 inches, 5.5 inches, and 4 inches long, respectively. In a similar way, velocities in other units, as feet per

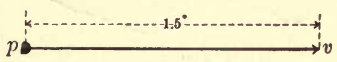

Fig. 10

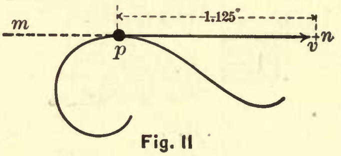

Fig. 11

minute, miles per hour, etc., can be indicated to suitable scales. The velocity of the point p (Fig. 10) which has a motion of 300 feet per minute in a rectilinear path, is represented to a scale of 200 feet per minute to the inch, by the line p-v, 1.5 inches long. If the motion is in any other path than a right line, the velocity at any position may still be represented by a straight line tangent to the path at that position, for the direction of the curved path at any point coincides with the tangent at that point.

In Fig. 11, the velocity of the point p, moving in the curved path at the rate of 45 feet per second, is represented to a scale of 40 feet per second to the inch by the line p-v, 1.125 inches long, lying along the tangent to the path and through the given position of p.

This graphic representation of velocity is of the greatest importance, as it makes many solutions possible on the drawing-board without the use of calculations; giving the results

required directly in connection with the regular process of designing, and permitting the easy determinations of results that could only be arrived at otherwise by tedious algebraic methods.

As to the accuracy of these graphic methods, it may be said that they are as close as can be used in a drawing itself; so, for the ordinary purposes of designing, they are all that can be desired in this respect. Furthermore, the graphic method has the advantage of showing a number of connected quantities in their true relation, appealing to the mind through the eye much more effectively than do numerical quantities. A limited experience with such problems as follow in this work will impress upon one the value of this method.

15. Newton's Laws of Motion.—Starting with the statement of *Newton's Laws,* which enunciate fundamental relations between force and motion, and with the familiar *Parallelogram of Forces,* we can readily develop the theory of the very important subject of *Resolution and Composition of Motions.*

Newton's Laws.

I.—Any material point acted upon by no force, or by a system of balanced forces, maintains its condition as to rest or motion; if at rest it remains at rest; if in motion it moves uniformly in a right line.

II.—Any material point acted upon by a single force, or by a system of unbalanced forces, has an acceleration of motion proportional to, and in the direction of the force, or the resultant of the system of forces.

III. Action and reaction are equal, opposite, and simultaneous.

16. Parallelogram of Forces.—The resultant of two or more forces applied at a point of a body is the single force which, if applied at the same point, will have the same effect on the body, as to rest or motion, as the given forces themselves. These forces which act together are called components of the single force, which is equivalent to their combined action.

Forces may be represented graphically, in a similar manner to that already explained in connection with the representation of

velocities; the direction of the line indicating the direction of the force, and the length of the line representing the magnitude of the force. If two forces, acting on a point, are represented in this way, the resultant of these forces is similarly represented by the diagonal of the parallelogram formed on the components as sides. For the proof of this, see Mechanics of Engineering, by Professor I. P. Church, page 4.

This proposition can be extended to cover the case of any number of forces acting at a point; for the resultant of any two of such a system of forces can be found, then the resultant of this first resultant (which exactly replaces the two original forces), and another of the forces can next be found, the resultant of this last resultant and another component can then be found, and so on till all of the original forces have been combined. The last resultant is the resultant of the system. By the reverse of the process just outlined, a single force can be replaced by two or more components.

The process of finding the resultant of several forces is called the *Composition of Forces;* the reverse process of finding the components of a force is called the *Resolution of Forces.*

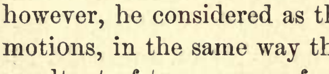

Fig. 12

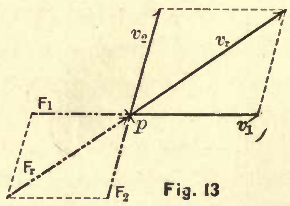

Fig. 13

17. Resolution and Composition of Motions and Velocities.—While a point may be acted on by any number of forces simultaneously, it can have but one *motion* at any time. This motion may, however, he considered as the resultant of two or more component motions, in the same way that any force may be considered as the resultant of two or more forces.

According to Newton's second law, if a point p (Fig. 12), which is initially at rest, is acted on by a single force F_1, the point will move in the direction of the force. The velocity at any instant may be represented to scale by v_1. If (Fig. 13) a second force F_2 acts simultaneously on p, the motion will be in the direction of the

resultant, F_r, of F_1 and F_2. This motion may then be considered as the resultant of two component motions in the directions of the respective component forces. Similarly, the velocity of p may be considered as the resultant of the velocities of the two component motions. Evidently this resultant velocity is the diagonal of the parallelogram of which the component velocities are adjacent sides.

From the preceding discussion, the following proposition can be drawn:

Parallelogram of Velocities.

If two component velocities of a point be represented, to scale, by the adjacent sides of a parallelogram, the diagonal of the parallelogram will represent the resultant velocities to the same scale.

Conversely, a velocity represented by a line, to scale, may be resolved into *any* pair of component velocities, which are represented to the same scale by the sides of a parallelogram of which the first line is the diagonal.

As in the case of forces, the reduction of more than two component velocities to one resultant can be effected by an extension of the above principles.

This method can be applied to any number of velocities, whether in one plane or otherwise. In Fig. 14 the resultant of v_1 and $v_2 = v_a$; the resultant of v_a and $v_3 = v_b$; the resultant of v_b and $v_4 = v_r$, = the resultant of the system.

In Fig. 15 the resultant of v_1 and $v_2 = v_a$; resultant of v_a and $v_3 = v_r$.

If the single velocity v_r (Figs. 13, 14, or 15) is given, it can be replaced by the velocities of which it is the resultant; for they, combined, are its equivalent.

Determining the resultant of a system of velocities is called *Composition of Velocities;* finding the components of given velocities is called *Resolution of Velocities.*

A system of velocities can have but one resultant; but a given velocity can have an infinite number of sets of components. The

velocity v (Fig. 16) may have for components v_1 and v_2; v_1' and v_2', or any number of sets of components; or the resolution is indefinite, because an infinite number of parallelograms can be drawn with the line v for a diagonal.

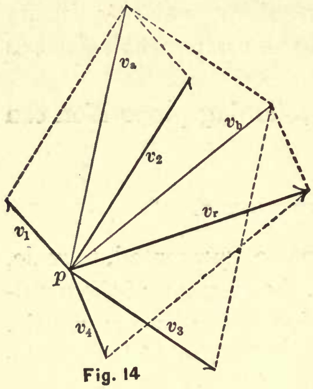

Fig. 14

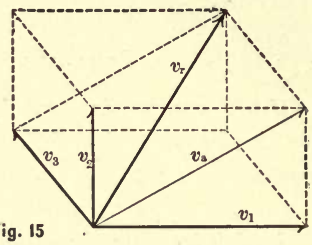

Fig. 15

If we know: (a) the direction of both components; (b) the magnitude of both; or (c) the magnitude and direction of one, there is a definite resolution [case (b) admits of a double solution]. For illustration of these three cases see Figs. 17, 18, and 19, respectively.

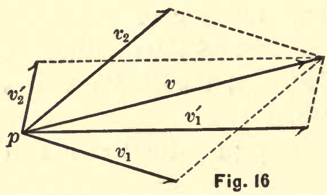

Fig. 16

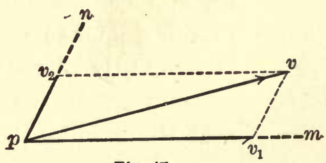

Fig. 17

Case (a). The given velocity p-v, Fig. 17, is to be resolved into components in the directions p-m and p-n.

From the point v draw line v-v_1 parallel to p-n, and cutting p-m in v_1; also from point v draw line v-v_2, parallel to p-m, cutting p-n in v_2; p-v_1 and p-v_2 are adjacent sides of a parallelogram meeting in p, and p-v is the diagonal of this parallelogram through this same point p, hence the velocities represented by p-v_1 and p-v_2 are the components of p-v in the given directions, p-m and p-n. It is evi-

dent that no other parallelogram can be formed on p-v as a diagonal with its sides in these given directions.

Case (*b*): The given velocity p-v, Fig. 18, is to be resolved into two components of the magnitudes indicated by m and n, directions to be determined.

With radius m and centre p draw the arc m_1-m_1', and with same radius and centre at v, draw the arc m_2-m_2'; also with radius n and centre at v draw the arc n_1-n_1', and with same radius and centre at

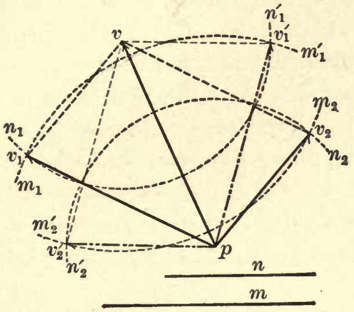

Fig. 18

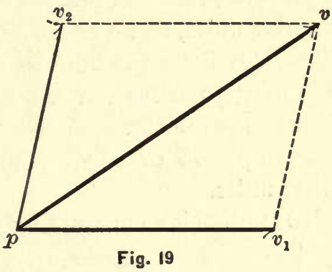

Fig. 19

p, draw the arc n_2n_2'; this gives four intersections. Connect these intersections with p by the lines p-v_1, p-v_2, p-v_1', and p-v_2'. By drawing lines from v to each of these intersections of the arcs, it is seen that *two* parallelograms are formed (p-v_1-v-v_2, and p-v_1'-v-v_2'), each having the given velocity p-v for a diagonal, with sides (p-v_1 and p-v_2, and p-v_1' and p-v_2', respectively) equal to the required components; hence there are two solutions to this case, both satisfying the condition that the velocity p-v be resolved into two components of values m and n.

Case (*c*) Fig. 19: The given velocity, p-v, is to be resolved into the component p-v_1, known as to magnitude and direction, and another component, entirely unknown.

Draw a line from v to v_1, also draw a line from v parallel to p-v_1, then draw a line from p parallel to v-v_1, cutting the line last drawn in v_2. p-v_2 is the required component; for the given component p-v_1 and this line last found form adjacent sides of a parallelogram with p-v, the given velocity, as a diagonal.

It will be seen from the preceding discussion, that in the resolution of a velocity into two components, or the composition of two velocities into one resultant, that there are six elements involved, viz.: the directions and magnitudes of three velocities, and that if four of these elements are known the other two may be determined, (except for the double solution of case *b*, in which two values satisfying the conditions are obtained).

The first case (*a*) is by far the most common in practical examples.

18. Angular Velocity.—When a point is revolving about some axis, permanently or temporarily, it is frequently convenient to express its rate of motion in angular rather than in linear measure. This rate of motion may be expressed in any system of time and angular units, as revolutions per minute or per second, degrees per second, radians per minute, etc. In many practical problems the rate of angular motion of a member is most conveniently stated in terms of revolutions per unit of time; but in analytical expressions the arc passed over by a point is often more readily measured in other units.

The radian is an arc of a length equal to the radius r; hence there are $\frac{2\pi r}{r} = 2\pi = 6.283$ radians to a circumference; or a radian is equivalent to $360° \div 6.283 = 57.3°$, nearly.

If a revolving point makes n revolutions about its axis per unit of time the space passed over in time unity, or its linear velocity, is $v = 2\pi rn$; and the angle traversed in the same time will be $\omega = \frac{2\pi rn}{r} = 2\pi n$ *radians.*

If the body A (Fig. 20) is revolving about the axis through C (which is perpendicular to the plane of the paper), at the rate of n revolutions per unit of time, the point p, at a distance r from the axis, has a linear velocity of $2\pi rn$; another point at p', at a distance r' from the axis, has at the same time a linear velocity $2\pi r'n$; and any two points at different distances from the axis have different linear velocities at any instant. But all points in the same rigid body when revolving about an axis must describe equal angles in the same time, and the angle (or arc) is being described at a rate, expressed in radians by $2\pi n$. This expression

for the rate of angular motion is what is called the *Angular Velocity* of the body; and it is necessarily the same at any instant for all points in the same rigid body. It will be noticed that the only variable in the expression for angular velocity as just derived is n, the number of revolutions per unit of time.

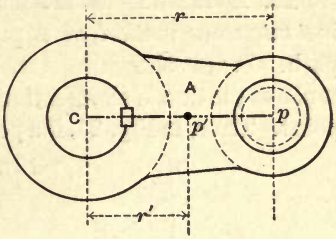

Fig. 20

Comparing the expression for angular velocity with that for the linear velocity of a revolving body, it is seen that it corresponds with the linear velocity of a point in the body *at a distance from the axis equal to unity*: from which we deduce the statement: The angular velocity of a body is numerically equal to the linear velocity of a point in the same body at unit distant from the axis.

The relation between the linear and the angular velocity of a point which is most frequently used and one that should be firmly fixed in the memory, is

$$\text{Angular velocity} = \frac{\text{Linear velocity}}{\text{Radius}}, \text{ or } \omega = \frac{v}{r}.$$

If a point revolves about a fixed centre with a linear velocity of 60 feet per second (720 inches per second), and with a constant radius of 18 inches (1.5 feet), its angular velocity is

$$\omega = \frac{60}{1.5} = 40 \text{ (radians per second)},$$

or

$$\omega = \frac{720}{18} = 40 \text{ (radians per second)}.$$

The space-units which measure the radius and the linear velocity must be the same, and the angular velocity is then expressed in radians per second, or per minute, according to whether the linear velocity time-units are seconds or minutes.

Angular velocity may be constant, or it may vary, uniformly or otherwise. If the radius remains constant, as in a body rotating about an axis to which it is rigidly connected, the angular velocity

must vary just as the linear velocity of any one point varies, as is seen from the above relation; or it varies directly as n.

19. Instantaneous Motion, Instant Centre, Instant Axis.—The *instantaneous motion* of a point is its motion at any point in its path. It was shown in Art. 14 that the direction of this instantaneous motion is along a tangent to the path at the position of the point. Thus, in Fig. 21, if a point is moving in the path m–n, the direction and velocity of its instantaneous motion when it occupies the position p, may be represented by the line p–v, tangent to m–n at p. This is equally true whether the point is moving in the path t–t', a–b, a'–b', a''–b'', or in any path whatever which is tangent to p–v at p. The instantaneous motion of a point is therefore independent of the *form* of its path. Motion in any of the possible paths is equivalent, for the instant, to rotation about c, c', or c'', or about any point in the line N–N', drawn through p perpendicular to p–v, for the path of a point having such rotation would be tangent to the other paths at p.

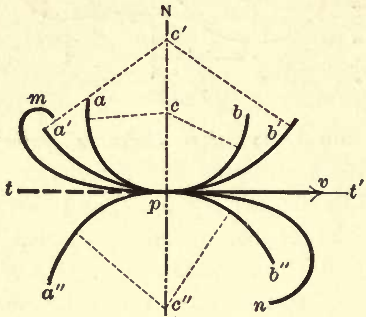

Fig. 21

An *Instantaneous Centre* (called more briefly an *Instant Centre*) af any plane motion of a body is a point about which the body may be considered as rotating at any instant relative to another body in the same plane. In Fig. 22 let p–v and p'–v' represent the velocities of the instantaneous motions of any two points, p and p' in the rigid body A, moving in the plane of the paper, and let p–n and p'–n' be perpendicular to p–v and p'–v' at p and p' respectively. Then the instantaneous motion of p is equivalent to rotation about some point in p–n as a centre. Likewise the motion of p' is equivalent to rotation about some point in p'–n'. Since A is a rigid body, p and p' can have no motion relative to each other, and the centre of rotation of p must also be the centre of rotation of p'. The point O, at the intersection of p–n and p'–n', is the only point that meets this requirement. It was shown

in Art. 10 that the plane motion of a body is determined by the motion of any two of its points not in the same perpendicular to the plane of motion. Therefore the instantaneous motion of A is equivalent to a rotation about O as a centre. In other words, O is an instant centre of the motion of A. This motion is assumed to be relative to the paper or any reference body in the plane of the paper. If the material of A and the reference body, B, are assumed to be extended to include O, Fig. 23, a pin could be put through O, materially connecting A and B and the instantaneous motion of A with reference to B would not be interfered with.

The instant centre of the relative motion of two bodies is a point

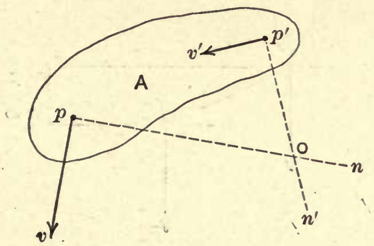

Fig. 22

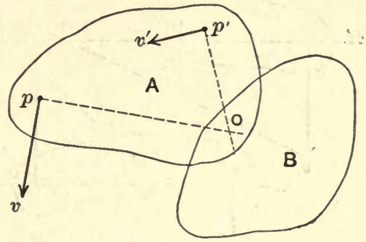

Fig. 23

at which they have no relative motion; it is the only point common to the two bodies for the instant.

It will be noted that the points p and p' may be moving in any paths whatever so long as these paths are tangent to p–v and p'–v' at p and p' respectively. Unless these paths are circular arcs having a common centre at O, the rotation of A about O does not continue for any finite time, which is implied when O is called an instant centre. If the paths of p and p' are such circular arcs, O is a *permanent centre* as well as an instant centre.

In general, it is only necessary to know the *direction* of motion of two points in a body having plane motion, in order to determine the location of the instant centre. When the points are moving in parallel paths special treatment is necessary. If the motions are at right angles to the line joining the two points,

as in Fig. 24, the instant centre lies somewhere on this line. The linear velocities of points in a rigid body being proportional to their distances from the centre of rotation, the location of the instant centre, O, may be found from the proportion $p-v:p'-v'::O-p:O-p'$, by the construction indicated. When, as in Fig. 25, the common direction of motion of the two points is not at right angles to the line joining them, the perpendiculars $p-n$ and $p'-n'$ are parallel, and their intersection, O, is at infinity. The instantaneous motion of the body is a rotation about a centre at infinity, or it is translation. In this case all points of the body have equal linear velocities.

It is more exact to refer to rotation or revolution about an

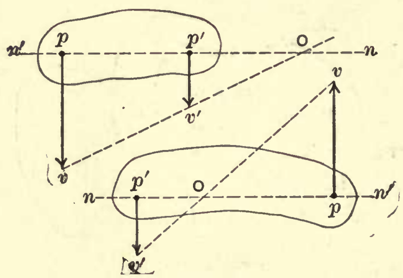

Fig. 24

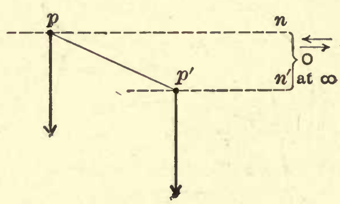

Fig. 25

axis than about a *centre.* In the case of plane motion the axis of rotation is always perpendicular to the plane of motion, and pierces every section of the body parallel to the plane of motion at its centre of rotation (Fig. 1). Since the motion of each section completely represents the motion of the whole body, it is customary in dealing with plane motions to refer to *instant centres* instead of the corresponding *instant axes.*

It was shown in Art. 10 that the motion of a body in space is determined by the motion of any three of its points not in the same right line. Such motion is at any instant equivalent to *rotation* about an axis *combined* with *translation* parallel to that axis. That is, it is a form of helical motion, as defined in Art. 11. In Fig. 26, the instantaneous velocities of any three points of a rigid body having any motion whatever in space are

represented by p–v, p'–v', and p''–v''.* In order that the instantaneous motion of the body may be equivalent to a rotation about some such axis as X–X', combined with a translation parallel to that axis, the components of p–v, p'–v', and p''–v'' perpendicular to X–X' must represent a plane rotation about X–X', and those parallel to X–X' must all be equal. That the axis X–X' can be located, and that the three instantaneous velocities can be resolved into such components is shown as follows:

From p_0 in Fig. 26(a), the lines p_0–v_0, p_0–v_0', and p_0–v_0'' are drawn respectively parallel and equal to p–v, p'–v', and p''–v''

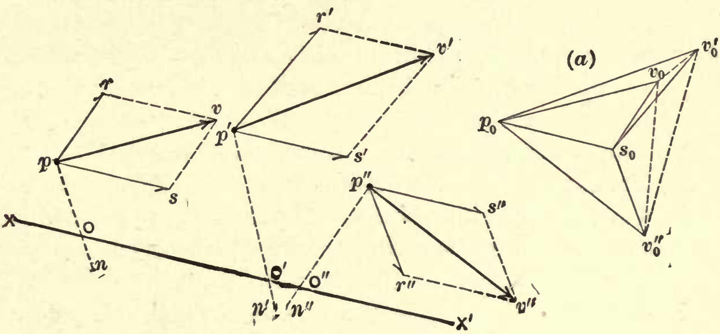

Fig. 26

in Fig. 26. The three points v_0, v_0', and v_0'' determine a plane. The line p_0–s_0 is drawn perpendicular to this plane, piercing it at s_0. The projections of p_0–v_0, p_0–v_0', and p_0–v_0'' on p_0–s_0 are all equal to p_0–s_0. The projections of the same lines on the plane perpendicular to p_0–s_0 are respectively s_0–v_0, s_0–v_0', and s_0–v_0''. These are all perpendicular to p_0–s_0'. In Fig. 26 the components of p–v, p'–v', and p''–v'' taken parallel to p_0–s_0 are p–s, p'–s', and p''–s''. These are all equal to p_0–s_0 and represent

* When these velocities are assumed it is necessary that the velocity of each point be so taken that its component along the line joining the point to each of the other points shall be equal to the component of the velocity of that point along the same line. Otherwise the motion would change the distance between the points, which is not consistent with the conception of a rigid body.

a translation of the body parallel to p_0–s_0. The components p–r, p'–r', and p''–r'' are respectively parallel and equal to s_0–v_0, s_0–v_0', s_0–v_0''. They represent plane motion of the body, since all are parallel to the plane perpendicular to p_0–s_0. Every plane motion of a body has been shown to be equivalent to an instantaneous rotation about an axis perpendicular to the planes of motion. The axis, X–X', of the rotation represented by p–r, p'–r', and p''–r'' must therefore be parallel to p_0–s_0. It pierces every plane section through the body perpendicular to p_0–s_0 in the instant centre of the motion of that section.

The motions ordinarily used in machinery may be considered as special cases of space motion. When the axis of rotation is fixed and there is a constant relation between the rotation and the translation, uniform helical motion results. When the translation is reduced to zero and the axis of rotation passes through a fixed point, the motion is spherical. Both these motions may be further reduced to plane motion as explained in Art. 13.

20. Free and Constrained Motion.—It follows from the statements of Art. 15 that if a point is to move in any prescribed path, the resultant of all forces acting upon the point in any of its positions must lie in a tangent to the path at the position of the point.* If the path be other than a straight line, this involves a constant change in the direction of the resultant force, caused either by a change in direction or magnitude (or both) of at least one of the components of this resultant. This is exactly what takes place in every such case; but the method of this readjustment of the resultant force affords the basis of a very important division of motions into two classes, viz.: *Free and Constrained Motions.*

A body which has no material connection with other bodies is called a free body; the planets are examples of this class of bodies. A planet revolves around the sun in a path or orbit determined by the resultant of all forces acting upon it; every disturbing action or force alters its path.

The motion of a body which has a material connection with another body, permitting motion relative to that body only in

* In this and the following discussion the point or body *must be considered as initially at rest* in each of the various positions it occupies in tracing its path.

certain restricted paths, is said to be constrained. The crank-pin of an engine has constrained motion. In this case, if motion takes place under the action of any force it must be in a fixed path, and no force, whatever its direction, short of one that will break or injure the machine, can cause motion in any other path.

The primary actuating force in the case of the crank-pin is the pressure or pull exerted upon the pin along the direction of the connecting-rod (neglecting frictional influence). This primary force does not, except at two instants in each revolution of the crank, act tangentially to the path of the body acted upon ; therefore we must look for some other force which combined with this primary force gives a resultant acting tangentially to the path of the crank-pin. While the study of such actions is not strictly within the province of the present treatise, it is important to clearly fix the nature of constrained motion, and for this purpose the distribution of force acting through the connecting-rod upon the crank-pin of the ordinary reciprocating steam-engine will now be briefly considered. Fig. 27 indicates the mechanism of the engine (without valve-gear). Figs. 28, 29, 30, 31, 32, 33, 34 show the connecting-rod and crank in different positions, or *phases.* The full lines indicate the directions of motion, and the dash-and-dot lines indicate forces acting.

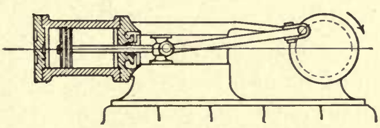

Fig. 27

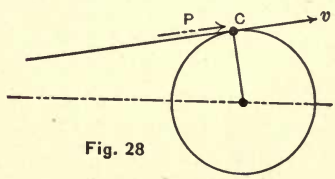

Fig. 28

In Fig. 28 the connecting-rod is at right angles to the crank, and therefore its centre line coincides with the tangent to the circle in which the crank-pin must move. As the force P, acting on the pin, is in the direction of the centre line of the rod, this force alone would produce motion in the prescribed path, and no other force need be considered as acting to produce such motion at this particular phase. The rod is under compression.

In Fig. 29 the condition is similar, except that the connecting-rod is now under tension instead of compression, and the action on

the pin is a pull instead of a thrust, but, as before, the force acts tangentially to the path.

In Fig. 30, however, the force P' exerted by the connecting-rod on the pin (thrust) is not in the direction of the tangent to the path, and hence it alone cannot produce motion in the required

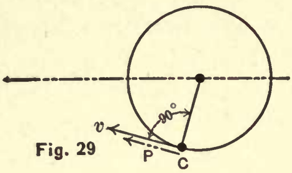

Fig. 29

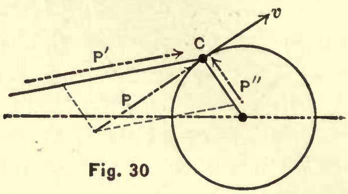

Fig. 30

direction. If, however, a force P'', be introduced in the direction of the centre line of the crank, of such magnitude that it, combined with P', will have a resultant P in the line of the tangent to the path of the pin, the conditions necessary to produce motion in the required direction will be present; and unless such a component of force is acting in conjunction with P', the required motion cannot take place. If the crank-pin were a free body this force would be an external force, but it will be seen that it would be very difficult to apply such an external force in the right direction and of the proper magnitude, for these requirements constantly change. In case of constrained motion the material connection (the crank in this case) supplies this force by its own resistance to a change of form. The primary acting force, alone, would impart motion in the direction of its own line of action, but this motion could not take place without changing the form of the crank, and the crank offers resistance to this change, by just the necessary amount for constrainment. As action and reaction are always equal, the force exerted on the crank to change its form is met by a corresponding counter-action, or reaction, just sufficient to give the required constraining force, and to cause motion in the circle of which the crank is the radius. This external force tending to change the form of the crank calls out within the material an internal molecular action known as stress, and this action is just equal to the external force. In this particular phase both connecting-rod and crank are subjected to compression.

In Fig. 31 the condition is similar to that of the preceding, except that the connecting-rod is under tension, the action on the pin

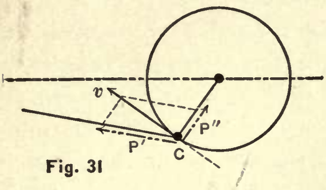

Fig. 31

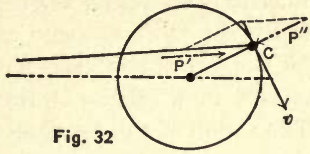

Fig. 32

is a pull, and the resistance of the crank is necessarily reversed, the stress now being a tension. This results of course in subjecting the crank to tension also, and as it is of a material that will resist this action, the motion in the required path is secured by the combined action of the force exerted upon the pin through the connecting-rod and of the secondary force called out in the crank. Figs. 32 and 33 show phases in which the stresses in the connecting-rod and crank are not similar.

When the crank-pin is at one of the "dead centres" *A* or *B*, as in Fig. 34, it will be noticed that the force exerted by the con-

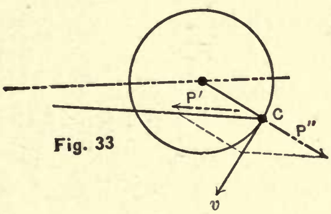

Fig. 33

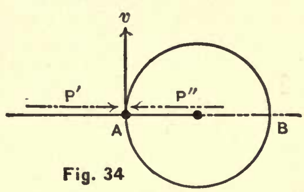

Fig. 34

necting-rod is at right angles to the direction of the pin's motion, and hence no force combined with it can give a resultant in the direction of the tangent to the path; the whole effect of this force, P', is now to compress or extend the crank (change its form), and none of it is available in moving the crank-pin. If it were not for the resistance of the crank at this time the pin would be impelled in the direction of P', at right angles to its proper path, but the resistance of the crank just balances the force received from the rod, and, according to Newton's laws, the pin is, at the instant, under a system of balanced forces, and, if in motion, it continues to move in a tangent to its path, unaffected by these forces except as they

influence friction. Of course this is only an instantaneous condition, and therefore the pin does not move through any finite distance under such a balanced system of forces.

Strictly speaking, the condition last considered is not equivalent to the action of no force at all, although the forces are balanced, for the pressure of the pin and of the shaft against the bearings results in a frictional resistance tending to retard the motion. The action of the fly-wheel also modifies the motion of the engine, reducing the fluctuation of velocity that would be experienced under the great variation of the resultant force throughout the revolution; but neither of these cases need be treated in connection with the present discussion, which is simply intended to exemplify the nature of constrained motion.

The distinguishing characteristic of a constrained motion is that, in a body having such motion, all points in the body have definite paths in which they move, if motion takes place under the action of any force whatever. The stresses produced in the restraining connections supply the components of force necessary to combine with the primary force, or forces, to give a resultant in the direction of the prescribed path. If these connections are strong enough to resist the maximum stress to which they are thus subjected, no farther attention is required to secure the proper adjustment of the resultant force to the prescribed path. The provision of the necessary strength is in the province of another branch of mechanics, and it may be assumed in the present work that such strength is provided.

It will be seen that absolute constrainment is not possible by the ordinary methods employed in machinery construction; because all materials are somewhat deformed under stress. Practical constrainment may always be secured; that is, the departure from the desired motion can be reduced to any required limit.

The *nature* of the constrainment depends upon the *form* of the constraining members. This is illustrated in Figs. 6, 7, 8 and 9, in which the nature of the relative motion of the parts is plainly determined by the form of the contact surfaces. The *degree* of constrainment is determined by the *dimensions* and *material* of the constraining members.

All motions used in machinery are either completely or partially constrained.

21. Mechanics is the science which treats of the relative motions and of the forces acting between bodies, solid, liquid, or gaseous.

"The laws or first principles of mechanics are the same for all bodies, celestial or terrestrial, natural or artificial." (Rankine.)

22. Mechanics of Machinery treats of the applications of those principles of pure mechanics involved in the design, construction, and operation of machinery.

Every problem of mechanics arising in connection with machinery is subject to the laws of pure mechanics, and we could conceive of its solution by the general methods of the larger science; but the operation would often be needlessly difficult, if not practically impossible, and more convenient special treatment has been developed for the limited class of phenomena connected with problems of mechanism. It has been seen that constrained motions are much more easily treated than are free motions; and all problems of motions of machines are included under constrained, or partially constrained, motions. It is mainly the distinction between free and constrained motions, in fact, that separates the Mechanics of Machinery from the more general science of Pure Mechanics.

23. A Mechanism, or train of mechanism, is a combination of resistant bodies for transmitting or modifying motion, so arranged that, in operation, the motion of any member involves definite, relative, constrained motion of the other members.

24. A Machine consists of one or more mechanisms for modifying energy derived from natural sources and adapting it to the performance of useful work. A machine may consist of a single mechanism according to this definition; but it has seemed best to make the following distinction between a mechanism and a machine: the primary function of the former is to modify motion; while that of the latter is to modify energy, and, of course, incidentally motion. The term "mechanism" becomes more general, and it includes the elements of a large class of instruments or apparatus, such as clockwork, engineers' instruments, models, and also most forms of governors, as well as some larger constructions, the function of which is essentially the modification of motion, and

which only do work incidentally, such as the overcoming of their own frictional resistance. There is a real and vital distinction between machines and such apparatus; but so far as a study of their motions is concerned, no such distinction need usually be made.

From the above definitions of mechanisms and machines, we may derive the following:

A Machine is a combination of resistant bodies for modifying energy and doing work, the members of which are so arranged that, in operation, the motion of any member involves definite, relative, constrained motion of the others.

The essential characteristics of a machine are:

(*a*) A combination of bodies.

(*b*) The members are resistant.

(*c*) Modification of energy (force and motion) and the performance of work.

(*d*) The motions of the members are constrained.

(*a*) A machine must consist of a combination of bodies.

The lever does not, by itself, constitute a machine, nor even a mechanism, and it only becomes such when combined with the proper fulcrum or bearing. Without this complementary member, properly placed and sustained, a definite, constrained motion is impossible.

The fulcrum is just as important an element in the make-up of the machine as is the lever itself. The screw is of no use in modifying motion or energy unless it is fitted with the proper envelope, usually called a nut. So with the wheel and axle. It makes no difference whether made from a single piece of material or built up from several pieces of stock, the wheel and axle is essentially one piece when completed, as there is no relative motion between the various parts, and it can only be of use in connection with appropriate supports or bearings. And so on with all other examples; the simplest machine must have at least two members, between which relative motion is possible.

(*b*) The members of a machine are generally rigid, but not necessarily so. Flexible belts, straps, chains, etc., confined fluids (liquid

or gaseous), and springs, often form important parts of machines. The flexible bands can only transmit force when subjected to tension; the confined fluids transmit force only under compression; springs may act under tension, compression, torsion, or flexure. These bodies are not rigid, in the usual sense of the word, but they are *resistant* under the particular action for which they are adapted; hence they can be used in special applications to great advantage. In fact their value in such applications is due to the absence of the property commonly designated as rigidity.

No material is absolutely rigid, and what is commonly and conveniently called a rigid body is one in which the distortions under load are so small as to be negligible for many purposes.

The action of springs, when carefully analyzed, is found to be identical in *quality* with that of the so-called rigid bodies. The characteristic of springs is the magnitude of the distortions. Every solid body possesses the property of yielding under a load to a greater or less degree, following the same general law as springs, within the safe working limit at least. The difference is one of degree only, but in this difference of degree lies the special fitness of springs for certain parts of machines.

(*c*) The machine is used to modify energy and do work.

It is interposed between some source of energy and the work to be done, and it adapts this energy, as supplied by or derived from natural sources, to the required work.

The conception of a machine involves the conception of some source of energy, an effect to be produced, and a train of mechanism suitably arranged to receive, modify and apply the energy derived from this source to the desired end.

The nature of the source of energy and of the work to be done determine the character of the machine, and the forms of the members for receiving the energy, transmitting, modifying, and applying it. The primary natural source of energy may be the muscular effort of animals, wind, water, heat (acting through such vehicles as steam, air, or other gases), etc. The secondary sources may be pulleys, gears, shafts, etc., deriving their energy, directly or indirectly, from some of the primary sources. The prime movers—windmills, water-wheels, heat-engines, etc.—are driven

from primary sources, while machinery of transmission—machine tools, dynamos, etc.—are actuated from secondary sources.

In a machine-shop, for example, the source of energy of the tools is the line-shaft, or the counter-shaft, according as the latter is, or is not, treated as a part of the tool; it evidently makes no difference how this shaft is driven, so far as the study of the individual tool is concerned. The source of energy being a rotating-shaft, the member of the machine receiving the energy must be a pulley, gear, sheave or other form capable of connection with such rotating-shaft. Energy may be transmitted by compressed air or by water under pressure; then the receiving member may be a piston, reciprocating in a suitable cylinder, or a wheel with appropriate vanes or blades attached.

In a similar way the desired result determines the motions and forms of the members producing it. When metal is to be planed a reciprocating motion is usually imparted to the member to which the piece operated upon is attached, or to the cutting tool. Thus, different classes of work, such as grinding grains or minerals, pumping water or other fluids, compressing air or other gases, weaving or spinning, cutting woods, stones, or metals, the transportation of materials, etc., each require an appropriate modification of the energy imparted to the receiving member of the machine.

In general, any of the sources of energy may be applied to produce any mechanical effect by means of proper trains of mechanism; and this gives rise to a very great number of possible machines. The working members of machines have been classified by Willis as:

(*a*) Parts receiving the energy.

(*b*) Parts transmitting and modifying the energy.

(*c*) Parts performing the required work.

To these might be added:

(*d*) Auxiliary parts, as regulators, etc.

(*e*) Frames for restraining the motions and sustaining the machines.

Various classifications of the parts of machines have been made

by different writers, but that of Willis has perhaps been most generally accepted. From a kinematic standpoint, such classifications are of doubtful value, and Reuleaux's masterly treatment of the subject indicates that all such divisions are artificial and arbitrary. This will be more fully discussed under Inversion of Mechanisms.

The working, or moving, members of a machine may be levers, arms, beams, cranks, cams, wheels with treads, blades, vanes, or buckets, with teeth or with flat or grooved rims, etc., screws and nuts, rods, shafts, links, and other rigid members; as well as belts, bands, ropes, chains (flexible members); and occasionally confined fluids, as water, oil, air, etc. Many modifications of these are used, and an indefinite variety of forms result, yet, kinematically, when reduced to the simplest forms, the variety of mechanisms is much less than would at first appear.

The frames which support the working parts and determine their motions are almost as varied in form and materials as the moving members themselves, but are capable of similar simple treatment. In fact, as will appear later, the frame may be treated as exactly equivalent to any other member of the machine, and so far as relative motion of the members is concerned, it matters not which particular piece is made "stationary."

The leading distinction between a machine and a "structure" (such as a bridge) is that the former serves to modify and transmit *energy*, or force and motion; while the latter modifies and transmits force only. Some parts of machines, as the fixed frames, are properly structures, while as a whole the construction is a machine.

(*d*) The relative motions of the members of a machine are constrained, or restricted to certain definite predetermined paths, in which they must move, if they move at all relative to each other.

The nature of constrained motion has been considered in Art. 20.

The leading characteristic of a mechanism or a machine is the constrainment of its motion. A structure does not permit relative motion of its members, or at most it only allows the very limited incidental motions, due to deformation of its members under loads, the effect of changes of temperature, etc. Occasionally what are

usually classed as structures, or parts of structures, do have prescribed motions, as the draw of a bridge, or a turn-table. These are not properly machines, but they do come under the preceding definition of a mechanism.

All artificial combinations of bodies, to which may be given the general name of constructions, and which have constrained motion, may be classed as mechanisms; and if intended to be employed in the performance of useful work, as machines.

The constrainment in mechanisms is sometimes partial, or incomplete. Thus in the case of a crane, in which the load is suspended by a chain or cable, the slightest horizontal force will sway the load-hook, and therefore change its path from the right line perpendicular to the earth's surface in which it normally moves. This does not affect the useful operation of the crane, as the hook is constrained against all undesirable motion, and for practical purposes the action is just as good as if the constrainment were complete. Often, in fact, this degree of freedom of motion is desirable.

Again, consider the familiar fly-ball (conical-pendulum) governor (Fig. 9) as used on many classes of steam-engines. The balls of the governor are constrained to the extent that their centres always lie in the surface of a certain sphere, that is, they have spherical motion; but they may move in any path whatever (within the limits of action), lying in the prescribed spheres. The path in which they actually travel depends upon the relations between their mass, angular velocity, and radius relative to the axis about which they revolve at the instant under consideration, and their motion can be determined only in connection with these forces.

In all but a few such cases as the latter we may study constrained motion quite apart from the forces involved in the operation of the mechanism.

25. Machine Design; Kinematics.—The design of a new machine or the analysis of an existing machine divides itself naturally into two quite distinct processes, as will appear upon brief reflection. In every machine energy is supplied from some source and so modified as to produce some useful effect. A train of

mechanism is employed to secure the required transfer or modification, and this intermediate mechanism must be adapted, first, to secure the *motion* demanded to produce the desired result ; and second, to transmit the necessary *force* without breakage or undue distortion of the members of the machine. It will readily appear that the motion system can often be planned or studied without considering the magnitude of the forces transmitted. As an illustration, consider the lever of Fig. 35, in which the distance from the fulcrum, O, to A is, say 1 foot, and distance from O to B is 2 feet. Now if B be moved through a distance of 2″, A will move through 1″. Suppose the resistance to act at A, and the force which overcomes it at B. The resistance at A may be 1 pound, 1 ton, or of any other amount whatsoever, then the force required at B to overcome it will be of a corresponding magnitude; but in any case the *ratio of the motions* or the velocity ratio of A to B will be determined by the length of the lever-arms, independent of the actual forces involved. Furthermore, if B moves 2″ in one second, A will move 1 inch in one second; if B moves 6″ in one second, A will move 3″ in one second; or if B makes 4 strokes per second, A will also make 4 strokes per second; but the (total) path described by A, or distance moved through, will be but half the path of B. So for any motion whatever of B, A will have a definite, corresponding motion, and the *ratio* of the motions will remain invariably the same, whatever the *actual* motion and the forces acting may be. It appears then that the ratio of the motions which A and B have, relative to the fixed member, is determined purely by geometrical considerations, and may be studied without taking into account anything else.

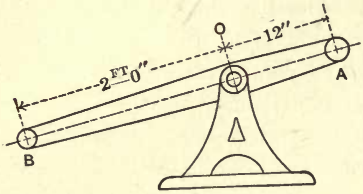

Fig. 35

The lever must not only give a required motion to one point for a given motion of another, but it must transmit a certain force; and having satisfied the motion requirement, it is necessary to give the lever, the pivot, and all parts subjected to load, sufficient strength to safely carry the loads. This second operation requires a knowledge and application of the physical properties of the mate-

rials used, and of other laws of mechanics than those relating to simple motion.

A similar discussion would apply to all ordinary mechanisms, but the foregoing is perhaps sufficient for the present purpose, viz., showing how the design of a machine may be taken up as two distinct processes, one of which can be completed, subject to certain modifications, before the other is considered.

Frequently the *actual* motions, velocities, and accelerations, as well as the external loading, are involved in the second process; for the weight of a part and the changes in direction and velocity of its motion produce stress in the restraining members. An example is the stress due to "centrifugal force." In the complete design of a machine, there are many other considerations affecting the durability, freedom from frictional and other losses, etc., that are no less important than the preceding, and all of these must be carefully weighed in their proper place; but these considerations are not within the province of the present work.

The two grand divisions of the Mechanics of Machinery outlined above are called:

(I) *The Geometry of Machinery, Pure Mechanism, or Kinematics;*

(II) *Constructive Mechanism, or Machine Design.*

In beginning the study of machinery, it is both logical and convenient to take up the above divisions, in the order given. The first division, Geometry of Machinery, Kinematics, or Machine Motions, will be the leading subject of the present work.

While, as in the illustration of the lever given above, the consideration of the forces acting is not, generally, involved in the study of machine motions, there are important classes of mechanisms the motions of which cannot be treated without taking cognizance of certain forces. Examples of these are the centrifugal governors, already mentioned, so commonly used on steam-engines and other motors, in which centrifugal force and the force exerted by springs, or by gravity, can not be separated from a treatment of the motions; also escapements such as are used in clocks and watches.

CHAPTER II.

GENERAL METHODS OF TRANSMITTING MOTION IN MACHINES.

26. Transmission through Space without Material Connection.—In most mechanisms motion is transmitted and controlled through actual contact of members of the mechanism; but there are certain exceptions as, for example: electric motors, escapements of clocks, governors, etc. The armature of the motor is caused to rotate by electromagnetic forces acting across an open space; the pendulum or balance-wheel of the clock-work is driven by the intermittent action of gravity or a spring, though the resultant motion is affected by the length of the pendulum or proportions of the wheel, independently of the intermittent connection with the escape-wheel; and the motion of the governor-balls is determined by the combined effect of centrifugal force and of gravity or of springs. In these instances, the motion of the members is not fully constrained, and the motions of such mechanisms cannot be treated by purely geometric methods, independently of the forces involved in the actual operation. With the exceptions of these and similar mechanisms, motion is only transmitted by direct contact of one material body with another. We are at present only concerned with such cases as the latter.

27. Transmission by Actual Contact.—These cases may be conveniently treated under two divisions; and the second division may be subdivided into two classes.

In every mechanism we have one member,—frequently called a link, whatever its form—driving another link; the former is called the driver, and the latter the follower.

The driver may have a surface which bears directly upon the follower, or there may be an intermediate piece serving to transmit the force and motion. This intermediate connector may be a rigid

bar or block; it may be a flexible band (as a belt, cord, or chain); or it may be a confined fluid column.

These various modes of connection give rise to the following classification:

- Modes of Transmission of Motion.
  - Direct Contact.
  - Intermediate Connectors
    - Rigid Links.
    - Flexible Connectors
      - Bands, etc.
      - Fluid Connectors.

28. Higher and Lower Pairing.—Figs. 36 and 37 represent

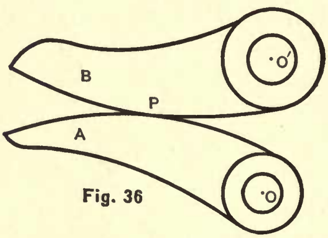

Fig. 36

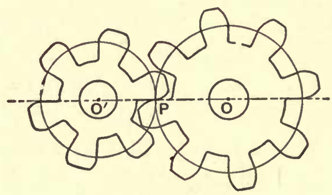

Fig. 37

examples of direct contact transmission, in which A will be considered as the driver and B as the follower. The contact in these examples is confined to a line (or a point), instead of being distributed over a finite surface. Such contact—line or point contact—constitutes what is termed *higher pairing;* while contact over a finite surface is called *lower pairing.* Higher pairing usually involves greater wear at the contact surfaces, and is generally to be avoided if it is possible to do so. The contact between the teeth of gears and that between most cams and their followers is necessarily higher pairing. However, there are many cases where it is perfectly practicable to introduce modifications in the construction which distribute the contact over a surface, without sacrifice of the kinematic relations. While this does not change the relative motion of driver and follower, it is practically advantageous in reducing the intensity of contact pressure, and consequently the wear of parts. It is usually desirable to substitute lower pairs for higher pairs where practicable. In certain cases,

where pure surface contact is not possible, a modification, which does not eliminate line contact, may be advantageously employed.

Figs. 38 and 39 show cases of transmission from the driver *A* to follower *B* by direct contact, higher pairing being used. In

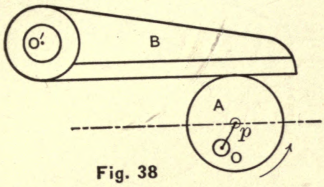

Fig. 38

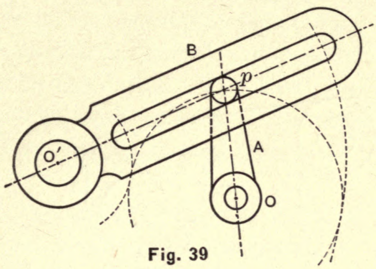

Fig. 39

these cases (Figs. 38 and 39), the kinematic action is the same as would result from contact between the point *p* of driver and the dotted "pitch-line" of the follower, as indicated by Figs. 40 and

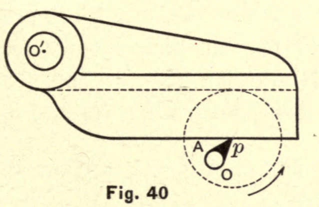

Fig. 40

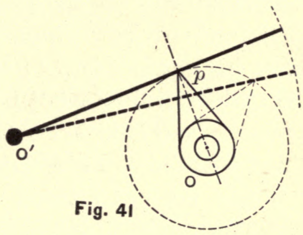

Fig. 41

41. These latter figures do not represent practical mechanisms, for of course it is necessary to have the contact parts of sensible size.

Figs. 42 and 43 show similar arrangements, each having a suitable block interposed between the driver and follower. These intermediate pieces do not change the transmission of motion in any degree, but it will be noticed that the driver now acts upon the block, and the block upon the follower, eliminating line contact entirely without sacrifice of the desired motion, and a better practical mechanism is thus obtained. The mechanisms of Figs. 38 and 39 are respectively identical, kinematically, with those of Figs. 42 and 43. An intermediate connector, or a new link, has been

introduced, and, in a sense, the mechanism comes under the second division in the above classification; but this intermediate connector, C, does not alter the transmission of motion. As we are not concerned in the least with the motion of this block itself, it

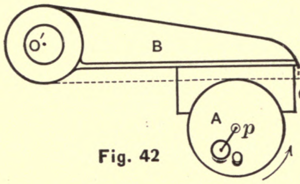

Fig. 42

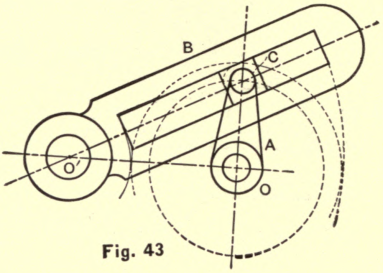

Fig. 43

may be neglected in the kinematic analysis, and such substituted mechanisms will be treated as direct contact under Division I. If desired, A could be treated as the driver of C; and C (which is the follower with regard to A) as the driver of B.

Except in cases where the contact surfaces of both links are either planes, surfaces of revolution, or regular helical surfaces, such substitution cannot be made; for these are the only contact surfaces possible in lower pairing. In gear-teeth, for example, it is not possible to avoid higher pairing.

There are other cases, as in cams, where it is practically of great advantage—even though line contact is not thus eliminated—to introduce an intermediate piece, replacing one kind of line contact by another kind. Thus, in Fig. 44, the cam could act directly upon the end of the rod B; but the friction would be excessive and the action would not be smooth, especially if the form of the cam departs much from that of a surface of revolution whose geometrical axis coincides with the axis of rotation. When a roller, C, of suitable size, is attached to the end of the rod much smoother action is obtained. The roll does not *rub* on the cam, as in the

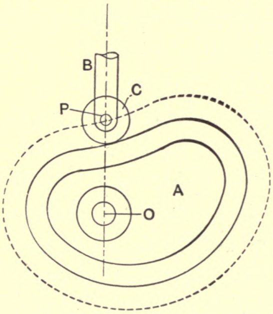

Fig. 44

direct sliding contact of the follower upon the driver, and the sliding action is transferred to the pin which carries the roll, where surface contact is procured. In this case, as in those of Figs. 42 and 43, the intermediate connector can be neglected kinematically. The motion transmitted to the follower corresponds to the contact of the centre of the pin, *p*, with a hypothetical surface—called the *pitch surface* of the cam—indicated by the dotted line. The relation of this pitch surface to the actual working surface, and the derivation of the latter from the former, will be treated later under the head of Cams.

In the following discussions of the angular velocity ratio of driver to follower, in direct-contact mechanisms, the auxiliary connector—the block, cam-roll, etc.—will be neglected, as it has been seen that its own motion is immaterial, and that it does not affect the velocity ratio of driver to follower.

It is interesting, in connection with the preceding discussion, to note that it is sometimes advantageous to employ line or point contact when the case will, kinematically, permit surface contact. The familiar roller-bearings and ball-bearings are examples. In these, friction and wear are reduced by the substitution of line or point contact for the ordinary surface-bearing, because by this substitution the grinding effect of sliding is replaced by a rolling of each member upon those with which it comes into contact.

29. Direct-contact Transmission.—The most general case of direct contact is between two surfaces such as are shown in Figs.

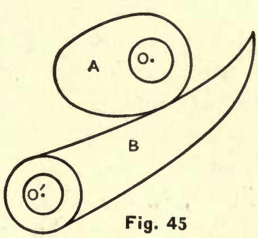

Fig. 45

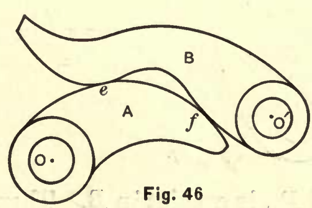

Fig. 46

36, 37, and 45. The surfaces may both be convex, or *one* may be concave, as in Fig. 45; but in the latter event the radius of curvature of the concave surface must always be at least as great as that of any portion of the other member that can come in contact with

it; otherwise a certain part of the concave surface will not come into contact, as in Fig. 46, between e and f, and the action will be discontinuous or irregular. Except for this limitation, the surfaces may be of any form; but the present discussion will be confined to those cases in which all the elements of the contact surfaces and both axes of rotation are parallel to each other. Members having such contact surfaces can have only plane motion relative to each other. There are special cases not coming under this class which will be treated later in the work.

In treating these plane motions the simplification referred to in Art. 10 can be applied, that is, the representation of these surfaces and their motions by their projection on a plane parallel to the plane of motion.

Motion can be transmitted by direct contact only by normal pressure between the surfaces. The action between the two parts in contact may have the nature of rolling, sliding, or mixed rolling and sliding. The last condition is the most general. The precise nature of these actions and the method of determining them will form the subject of a later section.

Referring to Fig. 47, it is evident that all points in A must rotate about O, and, likewise, all points in B must rotate about O'. Consequently the velocity of any point in either is represented by a line through that point perpendicular to the radius connecting it with its centre, O or O', as the case may be.

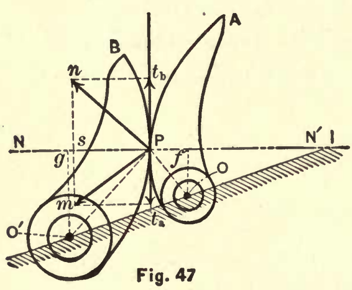

Fig. 47

The point of contact between the two members is at P, which may be regarded as the coincident position of two points, one of which, called P_a, is a point in A, while the other, called P_b, is a point in B. Then Pm and Pn represent the velocities of P_a and P_b, respectively.

The pressure between A and B is *transmitted in the direction of the common normal* to the two surfaces at the point of contact, and whatever the actual velocities of P_a and P_b, the *components* of

these two velocities *along the line of this normal* (NN') *must be equal* when they are in contact (as Ps). If the normal component of the velocity of P_b were greater than the normal component of the velocity of P_a, B would quit contact with A. On the other hand, if the normal component of the velocity of P_a is greater than that of P_b, A would enter the space occupied by B, and this is inconsistent with our conception of a rigid body.

As we are concerned only with the velocity ratio of the two members, and as this ratio is independent of the actual velocities, the velocity—either angular or linear—of one member may be assumed if not known, and this affords a means of determining the velocity of the other member at that instant. The angular velocity ratio of the driver to the follower is, in the general case, varying continually. Simple methods of determining this angular velocity ratio at any phase of the motion may be used, and a close analogy exists in these methods as applied to the three different classes of transmission. Each class will be discussed by itself, and the general relation will be deduced afterward.

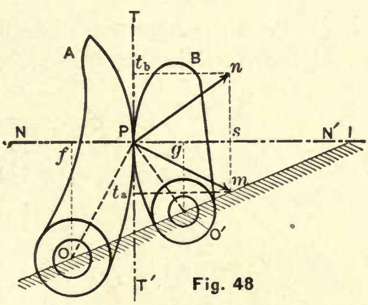

Fig. 48

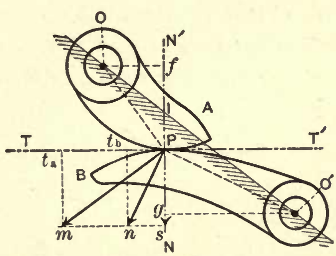

Fig. 49

If in Figs. 48 and 49, ω_1 = the angular velocity of A, be known, for the phase under consideration, the linear velocity of P_a can be found from the relation:

$$\text{ang. vel.} = \frac{\text{linear vel.}}{\text{radius.}}; \quad \text{or} \quad \omega_1 = \frac{Pm}{OP}.$$

Represent the linear velocity of P_a by Pm, and resolve it into its components along and perpendicular to the common normal NN'.

Thus the normal component Ps, and the tangential component Pt_a are obtained. The *direction* of the motion of P_b is known (perpendicular to PO'), and the normal component of its velocity must equal that of P_a, or it is Ps, hence the actual velocity of P_b, Pn, can be found (Art. 17, Case (a)); and its tangential component, Pt_b, may be found from this if desired. Having found in this way the linear velocity of P_b, its angular velocity, ω_2, may be obtained by dividing this quantity by the radius PO'; and the angular velocity ratio of A to B, for this phase of the motion $= \frac{\omega_1}{\omega_2}$, becomes known. A similar method could be employed in determining this ratio for any number of phases, and thus the motion of the follower, corresponding to the motion of the driver, whether uniform or otherwise, could be derived. The following demonstration establishes relations of the angular velocity ratio of driver and follower, in direct-contact mechanisms, which are much more expeditious and convenient in drawing-board practice.

In Figs. 48 and 49, let Pm and Pn represent the linear velocities of P_a and P_b, respectively. Drop perpendiculars Of and $O'g$ from O and O' upon the normal NN'. Ps is the common normal component of Pm and Pn. Pms and OPf are similar triangles; also Pns and $O'Pg$ are similar triangles.

$$\omega_1 = \text{angular velocity of } P_a \text{ about } O = \frac{Pm}{OP} = \frac{Ps}{Of}, \quad . \quad (1)$$

$$\omega_2 = \text{angular velocity of } B \text{ about } O' = \frac{Pn}{O'P} = \frac{Ps}{O'g}. \quad . \quad (2)$$

$$\frac{\omega_1}{\omega_2} = \frac{Ps}{Of} \times \frac{O'g}{Ps} = \frac{O'g}{Of}. \quad . \; . \; . \; . \; . \quad (3)$$

Prolong the normal and line of centres till they intersect at I; then IOf and $IO'g$ are similar triangles and

$$\frac{IO'}{IO} = \frac{O'g}{Of} = \frac{\omega_1}{\omega_2}. \quad . \; . \; . \; . \; . \; . \quad (4)$$

It follows from the above discussion that: *In direct-contact mechanisms the angular velocities of the members are, at any phase, inversely as the perpendiculars let fall from their fixed centres upon the line of the common normal of the two curves; or inversely as the segments into which the line of centres is divided by this normal.*

30. Link-connectors.—A relation very similar to that just derived can be deduced for the angular velocities of a driver and follower connected by a rigid link. In this case we are not concerned with the motion of the intermediate connector itself.

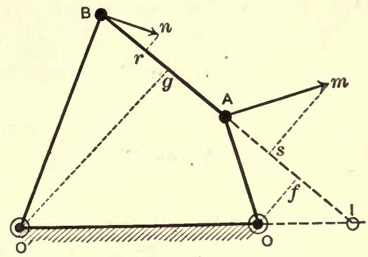

Fig. 50

Figs. 50 and 51 show two arms, OA and $O'B$, free to turn about the fixed centres O and O', and connected by the link AB. The velocity of the point A is represented by Am, perpendicular to OA. The velocity of B is shown by Bn, perpendicular to $O'B$; and its magnitude is determined by

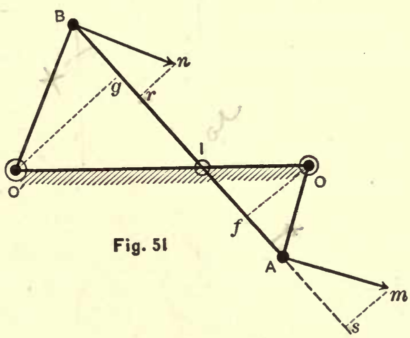

Fig. 51

the fact that its component in the direction of AB must equal the component of Am in this same line; for if these components are not equal, the distance between A and B must change, which is inconsistent with the conception of a rigid body; hence, if Am is assumed, Bn is thereby determined.

$$\text{Let } \omega_1 = \text{angular velocity of } A \text{ about } O = \frac{Am}{OA}, \quad . \quad . \quad (1)$$

$$\text{Let } \omega_2 = \text{angular velocity of } B \text{ about } O' = \frac{Bn}{O'B}. \quad . \quad (2)$$

Drop perpendiculars Of and $O'g$ upon AB; then triangles OAf and Ams are similar; also $O'Bg$ and Bnr are similar.

$$\text{From } OAf \text{ and } Ams, \frac{As}{Of} = \frac{Am}{OA} = \omega_1, \quad . \quad . \quad . \quad (3)$$

$$\text{From } O'Bg \text{ and } Bnr, \frac{Br}{O'g} = \frac{Bn}{O'B} = \omega_2, \quad . \quad . \quad . \quad (4)$$

$$\therefore \frac{\omega_1}{\omega_2} = \frac{As}{Of} \times \frac{O'g}{Br} = \frac{O'g}{Of} \text{ (as } As = Br). \quad . \quad . \quad . \quad (5)$$

Produce AB and OO' (if necessary) to intersect in I; then

$$\frac{IO'}{IO} = \frac{O'g}{Of} = \frac{\omega_1}{\omega_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From this reasoning the following statement is drawn:

In two arms connected by an intermediate link the angular velocities of the arms are to each other inversely as the perpendiculars let fall from the fixed centres upon the line of the link; or inversely as the segments into which the line of centres is cut by the line of the link (both of these lines produced, if necessary).

These relations may be seen from direct inspection, by assuming the system to be replaced by the two effective arms, Of and $O'g$, connected by the link fg. This new system would evidently be equivalent to the original system, for this particular position (but for no other); and as the arms Of and $O'g$ are perpendicular to the link, the linear velocities of f and g are equal; hence the angular velocities of the arms are inversely as the radii, or $\frac{\omega_1}{\omega_2} = \frac{O'g}{Of}$. This would apply to any phase; but the substituted arms, Of and $O'g$, would not be of the same lengths for different phases.

The relation deduced above may be arrived at also by means of the method of instantaneous axes. In Figs. 52 and 53, ω_1 and ω_2

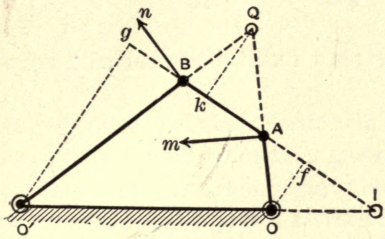

Fig. 52

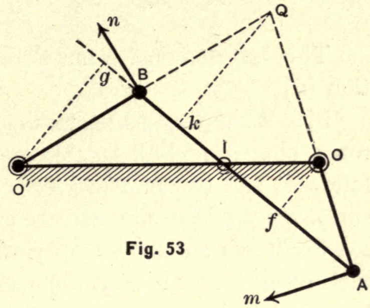

Fig. 53

have the same signification as before, and ω = angular velocity of the *connecting-link* about its instant centre.

A and B are two points in the connector AB; therefore the motions of these two points completely determine the motion of this link. The velocity of A is Am, perpendicular to OA; and the velocity of B is Bn, perpendicular to $O'B$. Therefore Q, at the intersection of OA and $O'B$, is the instant centre for the link AB in the phase shown (see Art. 19). As the angular velocity equals the linear velocity divided by the radius:

$$\omega = \frac{Am}{QA} = \frac{Bn}{QB}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Drop perpendiculars Qk, Of, and $O'g$ from Q, O, and O', respectively, upon the line of the link AB.

OAf and QAk are similar triangles; also $O'Bg$ and QBk are similar.

$$\therefore \quad \frac{\omega_1}{\omega} = \frac{Am}{OA} \times \frac{QA}{Am} = \frac{QA}{OA} = \frac{Qk}{Of}, \quad . \quad . \quad . \quad (8)$$

$$\frac{\omega}{\omega_2} = \frac{Bn}{QB} \times \frac{O'B}{Bn} = \frac{O'B}{QB} = \frac{O'g}{Qk}. \quad . \quad . \quad . \quad (9)$$

From equations (8) and (9),

$$\frac{\omega_1}{\omega} \times \frac{\omega}{\omega_2} = \frac{\omega_1}{\omega_2} = \frac{Qk}{Of} \times \frac{O'g}{Qk} = \frac{O'g}{Of} = \frac{IO'}{IO}. \quad . \quad . \quad . \qquad (10)$$

This last demonstration thus gives a result identical with equation (6).

31. Wrapping-connectors.—This term includes belts, bands, ropes, chains, and all flexible members used to connect a driver and follower, and transmitting force only under tension. The working surfaces may be of any convex cylindrical form; but concave forms are excluded, as the wrapper would not follow the depressions of such a form, and if used the action would not be smooth and continuous.

The term cylindrical as used above applies strictly in case of flat bands. In case of round cords, ropes, etc., the contact surface is usually grooved to correspond more or less closely to the form of the wrapper, but the motion is in this case equivalent to that which would be obtained by the neutral axis of the connector, wrapping upon an ideal pitch line of the member upon which it is carried (see Fig. 54). The mathematical (and kinematic) wrapper, or the pitch line, is the line xxx.

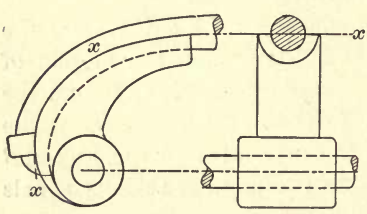

Fig. 54

In case of flat bands, also, the true pitch surface, and line of action, are at a distance from the physical face of the rigid member or carrier, equal to about one-half the thickness of the band. For the present purpose the connector will be treated as of no sensible thickness, and the surfaces shown in Figs. 55 to 58 are to be taken as the true pitch surface. The effect of thickness of connector will be discussed in a later chapter.

The band is flexible, but is supposed to be practically inextensible; and as it is subjected only to tension, the distance between any two points of the band cannot change. This implies that, whatever actual velocities two such points may have, the components

of these velocities in the direction of the connector must be the same at any instant.

In Figs. 55 and 56, a is the driver and b is the follower. Either

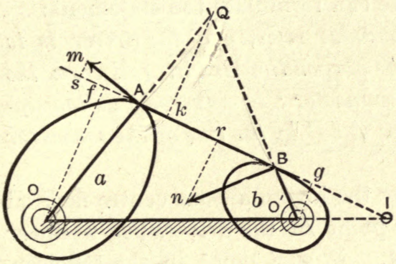

Fig. 55

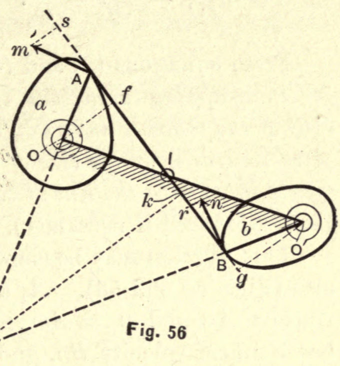

Fig. 56

of the tangent points of the band and carrier (A or B) is a coincident point of the band and of the member which it meets at that point; ω_1 and ω_2 are the angular velocities of the points A and B respectively, and Am and Bn are the corresponding linear velocities.

$$\omega_1 = \frac{Am}{OA}; \quad \omega_2 = \frac{Bn}{O'B}.$$

Since A and B are two points in the inextensible band, their components of velocity in the direction of the connector are equal, or $As = Br$. Drop perpendiculars from O and O' upon the line of the connector; then OAf and Ams are similar triangles; also $O'Bg$ and Bnr are similar.

$$\omega_1 = \text{angular velocity of } A \text{ about } O = \frac{Am}{OA} = \frac{As}{Of}, \quad . \quad (1)$$

$$\omega_2 = \text{angular velocity of } B \text{ about } O' = \frac{Bn}{O'B} = \frac{Br}{O'g}, \quad . \quad (2)$$

$$\therefore \frac{\omega_1}{\omega_2} = \frac{As}{Of} \times \frac{O'g}{Br} = \frac{O'g}{Of}, \quad (\text{as } As = Br). \quad . \quad . \quad . \quad . \quad (3)$$

Prolong the line of centres, OO', and the line of the band, AB, to meet in I; then IOf and $IO'g$ are similar triangles.

$$\therefore \frac{IO'}{IO} = \frac{O'g}{Of} = \frac{\omega_1}{\omega_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equations (3) and (4) we can formulate the statement:

In wrapping-connectors, the angular velocity of the driver is to that of the follower inversely as the perpendiculars let fall from the fixed centres upon the line of the connector; or inversely as the segments into which the line of centres is cut by the line of the connector (both produced if necessary).

This relation may be shown by the instantaneous centre method also (Figs. 55 and 56). A, as a point in the driver, has a linear velocity Am and $\omega_1 = Am \div OA$. B, as a point in the follower, has a linear velocity Bn, and $\omega_2 = Bn \div O'B$. The acting part of the connector, AB, has an angular velocity about Q of

$$\omega = \frac{Am}{QA} = \frac{Bn}{QB}.$$

Let fall Qk perpendicular to AB; then OAf and QAk are similar; also $O'Bg$ and QBk are similar.

$$\frac{\omega_1}{\omega} = \frac{Am}{OA} \times \frac{QA}{Am} = \frac{QA}{OA} = \frac{Qk}{Of}, \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{\omega}{\omega_2} = \frac{Bn}{QB} \times \frac{O'B}{Bn} = \frac{O'B}{QB} = \frac{O'g}{Qk}. \quad . \quad . \quad . \quad . \quad (6)$$

Multiply (5) by (6):

$$\frac{\omega_1}{\omega_2} = \frac{Qk}{Of} \times \frac{O'g}{Qk} = \frac{O'g}{Of} = \frac{IO'}{IO}. \quad . \quad . \quad . \quad . \quad (7)$$

This result accords with that of equation (4).

32. Similarity of Expressions for Angular Velocity Ratio in the Three Modes of Transmission.—By substituting the term *line of action* for "line of the normal," "line of the link," and "line of wrapping-connector," in the three cases of direct contact, link-con-

nector, and wrapping-connector, respectively; the following statement will apply to all of these modes of transmitting motion :

The angular velocities of the members are inversely as the perpendiculars let fall from their fixed centres upon the line of action; or inversely as the segments into which the line of action cuts the line of centres.

There are special cases in which the preceding theorems are not available, because the expressions become indeterminate; though these cases can be reconciled to the general statement. For example: see the direct-contact mechanism of Fig. 57, or the link-work of Fig. 58. In these mechanisms the centre about which B

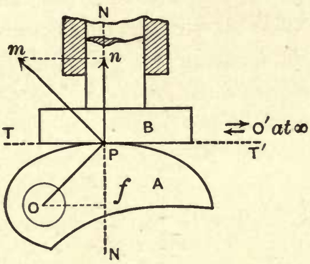

Fig. 57

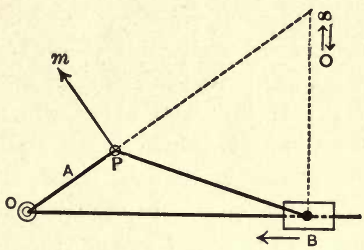

Fig. 58

rotates is at ∞, hence the perpendicular from it upon the normal $= \infty$ (also the segment from its centre to the line of action $= \infty$); and by the theorem of Art. 29; $\frac{\omega_1}{\omega_2} = \frac{\infty}{Of} = \infty$. This is consistent, as the angular velocity of the follower B, is o; hence that of the driver (A) is infinitely greater than that of the follower; but the result does not enable us to derive the linear velocity of the follower, for this equals the angular velocity multiplied by the radius, $= o \times \infty$, an indeterminate expression. The linear velocity of the follower is easily found by other means, however, as its normal component must equal that of the linear velocity of the driver, and the direction of the follower's motion is known. From this data, the linear velocity of the follower is derived (see Art. 17). A similar course of reasoning applies to the mechanism shown in Fig. 58.

In the linkwork shown by Fig. 59 the follower is not under control of the driver at the particular phase there represented. As

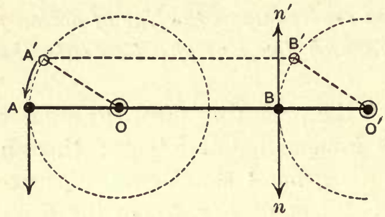

Fig. 59

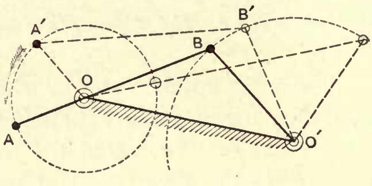

Fig. 60

A reaches the position shown (at either dead centre), it has no component of motion in the line of the link *AB*, hence there is no component compelling motion of *B*. As *A* passes this position, *B* might be moved in either of the two directions indicated by *Bn* or *Bn′*. If the shaft about which *B* rotates is provided with a fly-wheel, or similar device, its direction of motion may be maintained, and as soon as the dead centre is passed, *A* again exerts an influence over its motion. In the case of Fig. 60 *B* comes to rest as *A* passes the dead centre, *Bn* being zero at this phase.

33. Directional Relation.—It will be seen by reference to Figs. 48, 50, and 55 that the driver and follower both rotate in the same direction; that is, *both* members are turning in the right-handed, clockwise, or negative direction as angles are usually reckoned ; or *both* rotate in the reverse direction.* It will also be noticed that in the cases shown in Figs. 48, 50, and 55, the two fixed centres lie on the same side of the line of action.

In Figs. 49, 51, and 56, on the other hand, the fixed centres lie on opposite sides of the line of action, and the two members rotate in opposite directions; if one member has right-handed rotation, the other has left-handed rotation, and *vice versa*. From this observation of all of these general cases, the following statement in

* The follower is converted into the driver when such reversal takes place in Figs. 48 and 49, but not necessarily in Fig. 50. That this conversion does not take place in all direct contact and wrapping-connector mechanisms is evident from Figs. 39 and 204–209, in which either member may be the driver.

regard to the *Directional Relation* is quite evident: *In any of the three ordinary modes of transmission of motion the directions of rotation of the driver and follower are the same if the fixed centres of both lie on the same side of the line of action; and the directions are opposite if these centres lie on opposite sides of the line of action.*

34. Condition of Constant Angular Velocity Ratio.—It has been shown that with any of the three common methods of transmitting motion the angular velocities of the members are inversely as the segments into which the line of centres is cut by the line of action. Thus in any figure from 48 to 56 (except Fig. 54) $\omega_1 : \omega_2 :: IO' : IO$. If the angular velocity ratio is constant, $\frac{\omega_1}{\omega_2} = \frac{IO'}{IO} =$ a constant, and as OO'—the distance between the fixed centres—is a constant, I must be a fixed point in this line (or its extension) in order that the above condition be realized. Therefore it may be stated that: *The Condition of Constant Angular Velocity Ratio is that the line of action must always cut the line of centres (produced if necessary) in a fixed point.*

This condition is fulfilled by an infinite number of pairs of curves which may be used as the outlines of the acting faces of direct-contact members. It will appear later that the proposition just stated is of fundamental importance in the theory of teeth of gears.

The condition of constant velocity ratio is fulfilled in the case of wrapping-connectors when the driver and follower have faces which are right cylinders with the axes of the cylinders as the axes of revolution; for example, in the case of ordinary pulleys with crossed or open belts.

Constant velocity ratio is secured with link transmission when the driving and the driven arms are equal (Fig. 59), and the length of the connecting-link is equal to the distance between the fixed centres and parallel to the line of centres, as in the parallel rods of locomotives.

35. Nature of Rolling and Sliding.—When two pieces act together by direct contact they may roll upon each other, they may slide upon each other, or they may move relative to each other with a combined rolling and sliding action.

Fig. 61 shows two such members, in which p is the contact

point in the phase shown. If *r* and *s* are *any* two points which meet as the action continues (becoming coincident contact points), the arcs *pr* and *ps* must be equal if the action is pure rolling. If

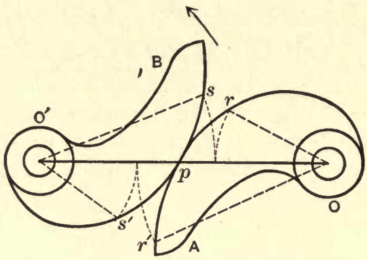

Fig. 61

for any increment of motion the corresponding arcs of action of the two curves are not equal, there must be some sliding between them. In pure rolling action no one point of either body comes in contact with two successive points of the other.

If a point of one of the bodies comes in contact with all successive points of the acting surface of the other (within the limits of its path), the action is purely sliding; for example, the piston in the cylinder of an engine.

In some cases, as in many cams and all gear-teeth, the action is mixed sliding and rolling. The sliding action must occur along the common tangent at the point of contact of the two surfaces.

36. Rate of Sliding and Condition of Pure Rolling.—It has been shown that in direct-contact mechanisms the normal components of the velocities of the points of contact must be equal. The tangential components may have any values, either in the same or in opposite directions. When the tangential components of the velocities of the contact points are in the *same direction and equal* there is no sliding, and the two velocities are identical as corresponding components are equal.

The rate of sliding is the difference of the tangential components if they are in the same direction, or their sum if they are in opposite directions; or: *The rate of sliding is the algebraic difference of the tangential components of the velocities of the points of contact.*

In direct-contact mechanisms the normal components of the velocities of the points of contact are always equal, and the tangential components are also equal when the action is pure rolling. Figs. 62 and 63 illustrate this condition, and the two velocities, *Pm*

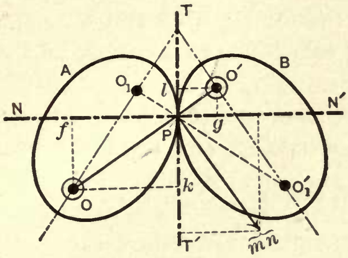

Fig. 62

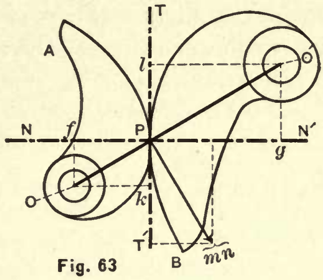

Fig. 63

and *Pn*, are identical. But *P*, as a point in *A*, moves at right angles to *OP*; and, as a point in *B*, it moves at right angles to *O′P*; therefore, when *Pm* coincides with *Pn*, *OP* and *O′P* are both perpendiculars to the same line at the same point and they must therefore lie in one right line. In order that this may occur, *P* must lie in the line of centres; or: *The condition of pure rolling is that the point of contact shall always lie in the line of centres.*

Any pair of direct-contact pieces bounded by curves which satisfy the condition just stated act upon each other with a pure rolling action; and any departure of the contact point from the line of centres is accompanied by sliding action. There are many sets of curves which may be employed to thus transmit motion by direct contact and with pure rolling action, among which may be mentioned: tangent circles (or circular arcs) rotating about their centres; pairs of equal ellipses each rotating about one of its foci with a distance between the fixed centres equal to the common major axis; and pairs of similar logarithmic spirals rotating about their foci.

As the common normal to two direct-contact members passes through the point of contact, and as this point always lies in the line of centres if the action is pure rolling, the common normal

cuts the line of centres in the contact point when pure rolling occurs. *The angular velocity ratio is inversely as the segments into which the line of centres is divided by the normal; or inversely as the perpendiculars let fall from the fixed centres upon the normal* (see p. 45, Art. 29). In pure rolling these segments are the contact radii themselves (OP and $O'P$ of Figs. 62 and 63), and therefore in such cases the angular velocities are inversely as the contact radii.

Drop perpendiculars, Ok and $O'l$, from the fixed centres (Figs. 62 and 63), upon the common tangent, TT', and it will be seen that the triangles OPk and $O'Pl$ are similar. $\therefore \dfrac{O'l}{Ok}=\dfrac{O'P}{OP}=\dfrac{\omega_1}{\omega_2}=$ the angular velocity ratio of the members. We may then use, if convenient, the following relation: *The angular velocity ratio, in direct-contact mechanisms having pure rolling action, is inversely as the perpendiculars from the fixed centres to the common tangent.*

In the circular-arc forms (Figs. 64 and 65) the perpendiculars from the centres to the tangent are the contact radii; thus the

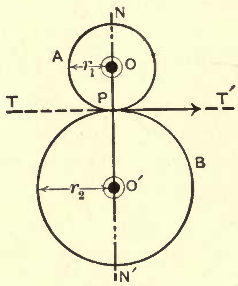

Fig. 64

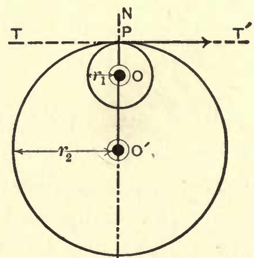

Fig. 65

well-known relation for tangential wheels of circular section,—that the angular velocities are inversely as the radii of the circles,—is seen to agree with the more general relations deduced in this article.

37. Constant-velocity Ratio and Pure Rolling Combined.— As stated in the preceding two articles, there are many pairs of curves which will satisfy either the condition of constant-velocity

ratio, or of pure rolling. There is but one class of curves, however,—viz., circular arcs rotating about their centres,—which can have at the same time *both* constant-velocity ratio and pure rolling (see Figs. 64 and 65). For constant-velocity ratio, the normal must cut the line of centres in a fixed point; for pure rolling, the contact point (through which the normal passes) must lie in the line of centres. If both of these requirements are met at the same time the contact point must be a fixed point in the line of centres; hence the contact radii must be constant; and therefore the outlines of the members are circular arcs.

38. Positive Driving.—Circular-arc members (right cylinders), as shown in Figs. 64 and 65, do not transmit motion positively. Actual physical bodies of the corresponding forms can transmit motion from one to the other only through frictional action. In the absence of friction, with such forms, no motion could be transmitted against any resistance; with friction a limited resistance can be overcome. There is no assurance that more or less slipping may not occur, and if this does take place the velocity ratio becomes both variable and uncertain.*

In such forms as those shown in Figs. 62 and 63, on the other hand, motion of the driver involves a positive and definite motion of the follower. It is now in order to determine the conditions necessary to insure positive or compulsory driving. It is sometimes stated that positive driving is only produced when the contact radius of the driver increases as the action proceeds ; thus in Fig. 61 A can only drive B positively when Op is greater than any preceding contact radius, as Or, and less than any succeeding radius as Or'. While this is the case with many forms, it is not a general

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* The tangential component of the velocity of either the driving or driven point represents its rate of sliding *along the tangent.* If the tangential components of the velocities of both these points are equal and in the same direction there is no sliding between them.

With *perfectly smooth* surfaces, one of the members could not move the other against the smallest resistance. In the practical cases where motion is transmitted by frictional action, the effectiveness increases as the departure from ideal smooth cylindrical surfaces becomes greater. Perfect cylinders, if such were possible, would not be of the slightest use in such cases.

requirement for positive driving. Figs. 66 and 67 show mechanisms in which A is the driver and B is the follower. It will be seen that A can rotate indefinitely, causing continuous rotation of B (though not with a uniform velocity ratio between A and B), and

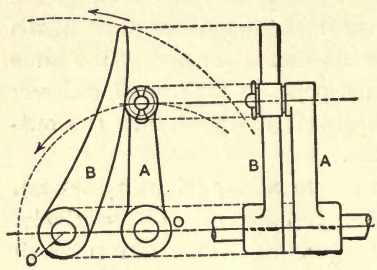

Fig. 66

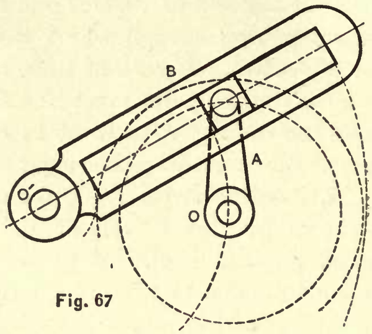

Fig. 67

at the end of each rotation the two members will return to the same relative positions they had at the start. It is evident then that the contact radius of the driver cannot increase indefinitely. It will be seen, also, that B may be the driver, and that a similar remark will apply in this case.*

Referring to Figs. 64, 65, and 68, it is seen that the motion, Pm, of the contact point of the driver lies in the direction of the common tangent, TT'; hence the normal component of this motion is zero. It is only the normal component of the driving point's motion which tends to impart positive motion to the follower ; and in the cases of Figs. 64, 65, and 68, where the motion of this point is wholly tangential and the normal component is zero, there is no tendency to produce positive driving. In other words, positive driving is assured only when the driving contact point has a component of motion in

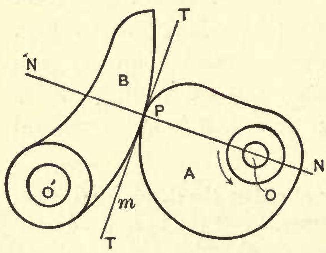

Fig. 68

* The presence of the intermediate roll or block is not essential, and the above statement would be equally true if A were simply provided with a pin engaging the follower.

the direction of the normal, and as the contact point moves perpendicularly to the contact radius, there can be no such normal component of motion when this radius is perpendicular to the common tangent; or, what is the same thing, when this radius coincides with the common normal. Positive driving cannot occur then if the common normal passes through the centre about which the driver rotates. The contact radius may coincide with the tangent, and in fact this is a very favorable position, as the motion of the contact point is then entirely in the direction of the normal, and there is no tendency to slide. If the common normal passes through the fixed centre about which the follower rotates the driver cannot impart positive motion to the follower; for in this position the normal component of the motion of the driven point is perpendicular to the path, and any motion in this direction is prohibited by the nature of constrained motion. A force directed toward the centre does not tend to produce rotation, but only to exert pressure against the bearings.

It is thus seen that positive driving cannot occur if the common normal passes through either of the fixed centres.

If the normal passes through the centre of the driver only, the driver can move, but motion is not transmitted to the follower. If the normal passes through the centre of the follower, the driver is locked, for its motion can have no normal component; but the follower may still move if under other influences, such as the action of a fly-wheel. If the common normal passes through *both* fixed centres, as in Figs. 64 and 65, the motions of both contact points are tangential and wholly independent, except for the frictional action.

In conclusion the following statement may be formulated: *The condition of positive driving is that the common normal shall not pass through the fixed centre of either the driver or the follower.*

39. Inversion of Mechanisms.—It was explained in Art. 5 that one body may have at any time distinct motions relative to different bodies, and that it is sometimes convenient to refer the motion of a member to some other part than the fixed frame. Take, for example, the crank and connecting-rod mechanism of the ordinary reciprocating-engine, as shown in Figs. 27 and 69. There are

four members of this mechanism: the crank, connecting-rod, cross-head and piston (the last two are kinematically one piece), and the frame (including the cylinder).*

In Fig. 69 these parts are designated by the letters *a, b, c,* and *d* respectively, and the shading of *d* is used to indicate that it is the stationary member. The crank, *a*, rotates about the centre O_{ad} relative to the frame, *d*, and this centre, O_{ad}, is the instant centre (also a permanent centre) for the motion of *a* relative to *d*. The frame, *d*, also rotates *relative to the crank, a,* about this same centre; for if we imagine the crank to be the fixed member, as in Fig. 70, *d* actually does rotate about this centre as the mechanism operates; but the change in the *relative* positions of the members is simply that due to the mode of constrainment, whichever member is fixed, and we have not changed the form of the mechanism in any way; hence the *relative motions* of the parts are the same under both conditions.

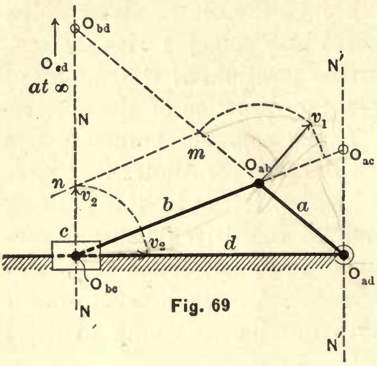

Fig. 69

The members in Fig. 70 are identical with those of Fig. 69, and their connections with each other are the same as before. If the member *b*, the original connecting-rod, be made the fixed member, as in Fig. 71, *a* and *d* are both moving members; but they still rotate, *relatively*, about O_{ad}. This mechanism, as shown in Fig. 71, is that of the oscillating steam-engine, in which *a* corresponds to the crank, *b* to the frame, *c* to the cylinder, and *d* to the piston-rod and piston. Fig. 72 represents another condition of this same mechanism, in which *c*, the original crosshead, is the fixed member. Under any of these four conditions the relative

* The notation used in the following discussion is this: small letters, *a, b, c,* etc., are used to designate the different members; *O* is used for all centres (instant or permanent), and the subscripts of *O* indicate the members which rotate relatively about it. Thus O_{ac} indicates that the member *a* rotates relative to *c* (or *c* relative to *a*) about the point designated as O_{ac}.

motion of a to d (or of d to a) is a simple rotation about the centre O_{ad}. In a similar way, the relative motion of a and b is a rotation about the centre O_{ab}; that of b and c is a rotation (or oscillation)

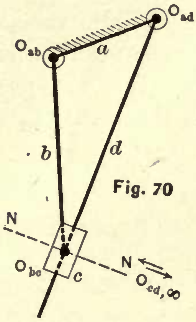

Fig. 70

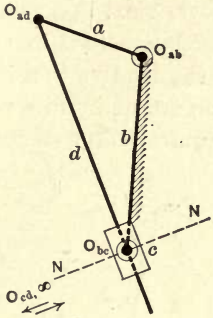

Fig. 71

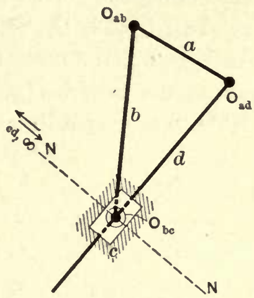

Fig. 72

about O_{bc}; that of c and d is a translation parallel to the centre line of d, or a rotation about a centre, O_{cd}, lying at infinity and in such a line as NN.

It is evident from the above illustration that apparently very different mechanisms may be essentially the same thing under different conditions. This was referred to in speaking of the classification of the parts of machines in Art. 24.

Such changes in the condition of a mechanism as are illustrated in Figs. 69, 70, 71, and 72, and which are effected by making different members correspond to the stationary part, or frame, are called by Professor Reuleaux *The Inversion of Mechanisms.*

Other examples of inversion will be given in a later part of this work.

40. Relative Motion between Different Members of a Mechanism.—The relative motions of the adjacent members of a mechanism are usually comparatively simple; thus in the mechanism of Fig. 73 the relative motions of a to b, b to c, c to d, and a to d are simply rotations or oscillations about permanent centres. The relative motions, in Fig. 69, of the corresponding adjacent members were traced in the preceding article.

In any mechanism each link (member) has a distinct motion relative to each of the other members. In four-piece mechanisms,

as those of Figs. 69 and 73, there are six distinct relative motions, viz.: *a* to *b*, *b* to *c*, *c* to *d*, *a* to *d*, *a* to *c*, and *b* to *d*. Each of these motions is equivalent to rotation or oscillation about a centre (permanent or instant). Four of these are permanent centres in the mechanisms referred to above, viz.: O_{ab}, O_{bc}, O_{cd}, and O_{ad}.*

In Fig. 73, O_{bc} rotates relative to *d* in an arc of finite radius. In Fig. 69, the motion of O_{bc} relative to *d* is equivalent to rotation about the centre O_{cd}, in an arc of infinite radius. If it were possible to supply a link of infinite radius connecting the point O_{cd} of

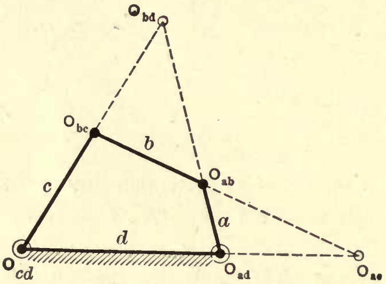

Fig. 73

d with the point O_{bc}, the mechanism of Fig. 69 would be similar in character to that of Fig. 73; or the former may be considered as a limiting form of the latter. The practical mechanism of Fig. 69 is the exact kinematic equivalent of such an imaginary mechanism.

The relative motions of the opposite links, *a* and *c*, and *b* and *d*, of Figs. 69 and 73, are not so evident as are the motions of the adjacent members; but the instant centres for these motions are readily located from the principles of Art. 19. In the first place it is to be noted that the instant centre for two members is a common or coincident point of both, for it is a point with regard to which neither of them has any motion. If this point lies

* In the mechanism of Fig. 69 the centre O_{cd} is at infinity. While it is not an actual physical pin like the other permanent centres, still it is properly a permanent centre rather than an instant centre, because it is equivalent to a fixed centre of a link of infinite length.

outside of either actual body, this body may be imagined as extended to include this centre, for a body may have rigid connection with any point relative to which it has no motion, as stated in Art. 5.

It is to be borne in mind, then, that O_{ab} is a point in both a and b; O_{ac} is a point in both a and c, etc. This enables us to locate the instant centres for the opposite links. In Fig. 73 O_{ab}, as a point in a rotating relative to d, must move in a line perpendicular to the line joining the points $O_{ad}-O_{ab}$; hence its motion is equivalent to a rotation about some point in this line or its extension (see Art. 19). Likewise, the motion of O_{bc} relative to d is equivalent to a rotation about some point in the line $O_{cd}-O_{bc}$ or its extension. But O_{ab} and O_{bc} are two points in the link b, and, as such, must have the same motion, i.e., rotation about the point O_{bd} at the intersection of the lines $O_{ad}-O_{ab}$ and $O_{cd}-O_{bd}$. The motion of these two points determines the motion of the link. Therefore O_{bd} is the instant centre for the motion of b relative to d. In a similar way, all points of b rotate relative to a about the centre O_{ab}, and all points of d rotate relative to a about O_{ad}. The points O_{bc} and O_{cd} are points in b and d, respectively; hence they move, relatively to a, perpendicularly to the lines $O_{bc}-O_{ab}$ and $O_{cd}-O_{ad}$ respectively, and the intersection of these lines, or O_{ac}, is their common centre of rotation relative to a. But O_{bc} and O_{cd} are two points in the link c; therefore the point O_{ac} is the instant centre for the motion of c relative to a. We have thus located all of the instant centres for the mechanism of Fig. 73.

Four of the centres for the mechanism of Fig. 69 have been located, viz.: the permanent centres O_{ad}, O_{ab}, O_{bc}, and O_{cd} (the last at infinity). The centres for the two pairs of opposite links, b and d, and a and c, are yet to be found. The former, O_{bd} is readily found; for O_{ab} (a point in b) moves perpendicularly to the centre line of a, and O_{bc} (also a point in b) moves perpendicularly to NN; therefore the intersection of these lines, or O_{bd}, is the required instant centre for the motion of b relative to d.

The reasoning by which the centre for the relative motion of a and c is found is somewhat more involved. The point O_{bc} is a point common to b and c. All points in b rotate relative to a about the

centre O_{ab}; therefore O_{bc} as a point in b rotates about this point, or it moves, relative to a, perpendicularly to the centre line of b (the line $O_{bc}-O_{ab}$). The point O_{cd} (at infinity) is a point common to c and d. As a point in d it rotates about O_{ad} relative to a, moving perpendicularly to a vertical line through O_{ad}, or to $N'N'$. One point of c (O_{bc}) moves perpendicularly to $O_{bc}-O_{ab}$, and another point of c (O_{cd}) moves perpendicularly to $N'N'$; hence the instant centre for the motion of c relative to a is at the intersection of these two lines, or at the point O_{ac}.

By considering c, instead of a, as the stationary member, the same result may be reached rather more easily.

It is possible to locate all of the instant centres of a mechanism of four members by the principles already given; but with higher numbers of members this cannot always be done, and a very important theorem given by Professor Kennedy affords a ready solution in these more difficult cases. This theorem is often advantageous even in four-link mechanisms, and by its aid the rather tedious reasoning employed above in finding O_{ac} for the mechanism of Fig. 69 can be avoided.

The statement of this theorem, as given by Professor Kennedy, is: *"If any three bodies a, b, and c have plane motion, their virtual [instant] centres O_{ab}, O_{bc}, and O_{ac} are three points upon one straight line."*

This theorem applies to *any three bodies* having plane motion, whether they be members of the same mechanism or not. Two of the three centres being known, or assumed, the following demonstration proves that no point lying outside of the line connecting the known centres can be the required third centre; hence this third centre must lie in the line connecting the other two, as stated in the theroem.

In Fig. 74, let a, b, and c be *any* three bodies moving in a plane, members of a single mechanism as indicated by the heavy lines, or entirely independent bodies.

Suppose that a rotates relative to b about the centre O_{ab}, and that c rotates relative to b, about the centre O_{bc}. Then O_{ab} is a point common to a and b, and O_{bc} is a point common to b and c. Assume that such a point as O' is the instant centre for the relative

motion of a and c; then this point is a common point of a and c. All points in a must rotate relative to b about O_{ab}, and all points in c must rotate relative to b about O_{bc}. As a point in a the motion of O' relative to b must be in a direction perpendicular to O_{ab}–O', while as a point in c its motion relative to b must be in a direction perpendicular to O_{bc}–O'. A point can have but one motion relative to a given body at any time, and therefore these perpendiculars must coincide. This is possible only when O_{ab}–O' and O_{bc}–O' are in one straight line, viz.: the line joining the given centre O_{ab} and O_{bc}. It is thus seen that the point O' cannot be the centre required, unless it does lie in such line.

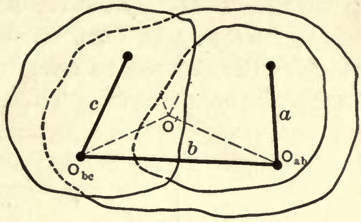

Fig. 74

This theorem does not locate the centre O_{ac} definitely; for it may be any place along the line of O_{ab}–O_{bc}, between these centres, or beyond either of them. This is as it should be, for in the arrangement of Fig. 74 there is no prescribed connection between a and c, and their relative motion is therefore not definitely constrained.*

If a fourth member, d, which will constrain the motion of a relative to c—such as a link connecting the free ends of a and c —be introduced, another combination of three members (as a, c, and d) may be taken, in which O_{ad} and O_{cd} are known, and, by the theorem, the other centre for this combination (O_{ac}) will lie in the line of the known centres. In this constrained four-link

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* It is to be noticed that the theorem discussed above has reference to three members, and that these three members involve three instant centres; any member has a centre with reference to each of the other members. In a combination of three bodies every letter which stands for one body is used twice as a subscript to O. If two of the three centres are given their symbols will have one common letter in their subscripts, and the third (required) centre will have for a subscript the two odd letters. Thus if O_{ab} and O_{bc} are the given centres, O_{ac} is the third. This is a convenient aid in applying the above theorem to a mechanism.

mechanism there are two lines, each of which contains O_{ac} (viz.: O_{ab}–O_{bc}, and O_{ad}–O_{cd}); hence O_{ac} is at their intersection, or its position is definitely determined (see Fig. 73).

By referring to Figs. 69 and 73 it will be seen that the locations of the centres, as already determined, agree with the statement of the theorem. Thus, as to the links a, b, and c, the

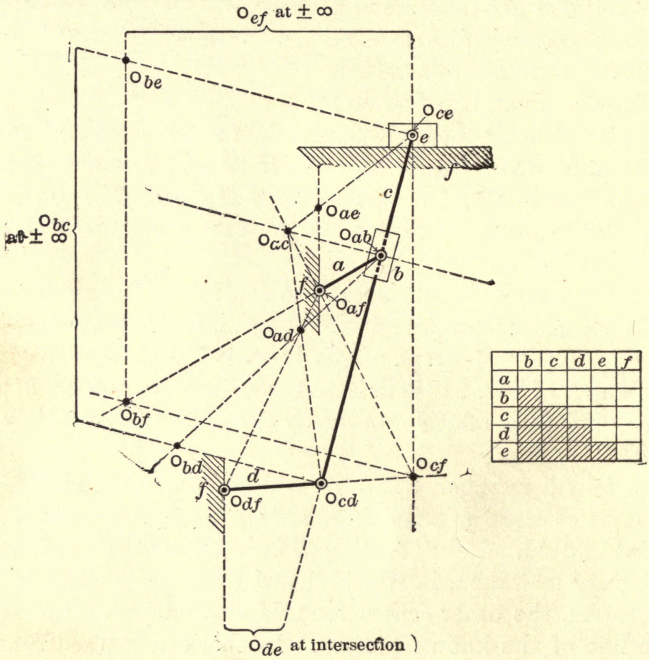

Fig. 75

centres O_{ab}, O_{bc}, and O_{ac} lie in one line; also, as to a, c, and d, O_{ac}, O_{ad}, and O_{cd} lie in one line, and O_{ac} lies at the intersection of these two lines.

As an illustration of the application of the above theorem to more than four members, the mechanism of a common type of crank shaper, indicated in Fig. 75 by what is called a skeleton drawing, may be taken.

In this machine there are six members: the crank, a; the sliding block, b; the vibrator, c; the link, d; the ram, e; and the frame, f, to which the members a and d are pivoted, and which is provided with guides for the motion of e. This mechanism has 15 instant centres,* of which 7 are also permanent centres. Of the permanent centres, O_{af}, O_{ab}, O_{cd}, O_{ce}, and O_{df} are located at the points of connection of adjacent members, while O_{bc} and O_{ef} are at infinity. The other centres are found by the use of Kennedy's theorem.†

The following scheme suggests the solution of this problem:

Centre Required.	Lies at Intersection of the Lines.
O_{cf}	$O_{cd} - O_{df}$ and $O_{ce} - O_{ef}$
O_{de}	$O_{cd} - O_{ce}$ " $O_{df} - O_{ef}$
O_{ac}	$O_{ab} - O_{bc}$ " $O_{cf} - O_{af}$
O_{bf}	$O_{ab} - O_{af}$ " $O_{bc} - O_{cf}$
O_{ae}	$O_{ac} - O_{ce}$ " $O_{af} - O_{ef}$
O_{ad}	$O_{af} - O_{df}$ " $O_{ac} - O_{cd}$
O_{bd}	$O_{ab} - O_{ad}$ " $O_{cd} - O_{bc}$
O_{be}	$O_{bf} - O_{ef}$ " $O_{bc} - O_{ce}$

The method of instant centres will be frequently used in the later part of this work, especially in treating linkwork; but it may be well to give an illustration of its use at the present place. It is to be remembered that the linear velocity of a point which is moving relative to any body is proportional to its distance from the centre about which it rotates relative to that body. In Fig. 69, for example, the body a rotates relative to the stationary member,

* In a mechanism of n members, there are $\frac{n(n-1)}{2}$ instant centres, some of which are also permanent centres.

† A diagram such as is shown in connection with Fig. 75 is convenient in this work. The unshaded spaces indicate the centres to be located, and the memory is aided by checking off, in the proper place, each centre as it is found.

d (the frame), about O_{ad}. If the linear velocity, v_1, of O_{ab} is known, the linear velocity, v_2, of the crosshead (piston) is readily found by the principles of instant centres. The point O_{bc} is common to the connecting-rod, b, and the crosshead, c; while the point O_{ab} is common to the crank, a, and the connecting-rod, b. The instant centre of b relative to d is O_{bd}; then, as all points of b must have the same angular velocity about O_{bd} their linear velocities are proportional to their distances from this centre ; hence

$$v_2 : v_1 :: O_{bd}\text{–}O_{bc} : O_{bd}\text{–}O_{ab}.$$

If v_1 be laid off from O_{ab} toward O_{bd}, along the line connecting these points, and then a line, mn, parallel to the connecting-rod, be drawn till it cuts the normal NN, the length on this normal from O_{bc} to n equals v_2, from the above proportion. As the motion of c relative to d is a translation, all points of c have the same velocity relative to d ; hence v_2 is the velocity of the crosshead, or piston, relative to the frame, or cylinder.

In an engine the crank rotates about the shaft with a velocity which is usually taken as uniform ; while the velocity of the cross-head (or piston) is variable. The velocity of the piston can be found for any phase by laying off the crank-pin velocity along the extension of the crank, drawing a line (as mn, Fig. 69) parallel to the connecting-rod till it cuts the normal (NN) through the cross-head pin.

A modification of the preceding construction is often even more convenient. Lay off the line N'–N' (Fig. 69) through the centre of the shaft, O_{ad}, perpendicular to the line of the piston travel. The connecting-rod (extended if necessary) cuts N'–N' in the point O_{ac}, then, since $v_2 : v_1 :: O_{bd}\text{–}O_{bc} : O_{bd}\text{–}O_{ab}$ and, from similar triangles, $O_{bd}\text{–}O_{bc} : O_{bd}\text{–}O_{ab} :: O_{ad}\text{–}O_{ac} : O_{ad}\text{–}O_{ab}$, it follows that $v_2 : v_1 :: O_{ad}\text{–}O_{ac} :: O_{ad}\text{–}O_{ab}$.

It appears from this proportion that when the length of the crank, $O_{ad}\text{–}O_{ab}$, is taken to represent the uniform crank-pin velocity, the cross-head velocity is represented by the distance, $O_{ad}\text{–}O_{ac}$, the intercept on the perpendicular, N'–N', between

the shaft centre and the line of the connecting-rod, the latter extended if necessary.

This relation is also evident from the consideration that the instant centre, O_{ac}, as a common point of a and c, has the same velocity and direction of motion in c as in a. As a point of c the linear velocity of O_{ac} is v_2, since all points of c have the same velocity of translation. This velocity is found from the motion of O_{ac} as a point in a by the proportion $v_2 : v_1 : : O_{ad}–O_{ac} : O_{ad}–O_{ab}$.

In general, when the instant centres, O_{ab}, O_{ac}, and O_{bc}, for the plane motion of any two bodies, a and b, relative to each other and to a third (reference) member, c, are located, the linear velocity of any point in b corresponding to a given linear velocity

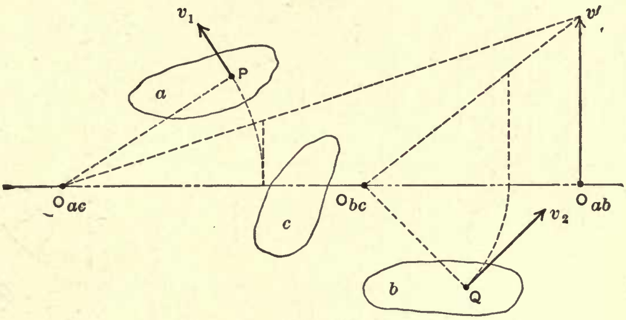

Fig. 75*a*

of any point in a can be found graphically. In Fig. 75*a*, let v_1 represent the given linear velocity of any point, P, in a, and let the corresponding velocity, v_2, of any point, Q, in b, be required. The linear velocity, v', of O_{ab} as a point of a is found from the proportion—$v' : v_1 : : O_{ac}–O_{ab} : O_{ac}–P$, by the construction shown. Using O_{ab} as a point of b, the corresponding linear velocity of Q is found, by a similar construction, from the proportion, $v_2 : v' : : O_{bc}–Q : O_{bc}–O_{ab}$.

The ratio of the angular velocities of any two bodies, a and b, having plane motion relative to a third body, c, may also be determined when the three instant centres are located. Let ω_1 and ω_2 be the respective angular velocities of a and b relative to c in Fig. 75*a*. Since O_{ab} as a common point of a and b, has a

linear velocity v', the angular velocities of a and b relative to c are equal to this linear velocity divided by the respective instant radii. That is, $\omega_1 = v' \div O_{ab}–O_{ac}$ and $\omega_2 = v' \div O_{ab}–O_{bc}$. Hence $\omega_1 : \omega_2 :: O_{ab}–O_{bc} : O_{ab}–O_{ac}$. When this proportion is used in the case of any mechanism the resulting value of the angular velocity ratio is identical with that obtained by the methods of Arts. 29–32.

41. Velocity Diagrams.—It has been shown in the preceding article how the method of instant centres can be used to determine the linear velocity of one point from the known velocity of another point. It is often desirable to represent, graphically, the velocities of a point at various phases of a mechanism, and this is done conveniently by velocity diagrams. Fig. 76 shows the mechanism of

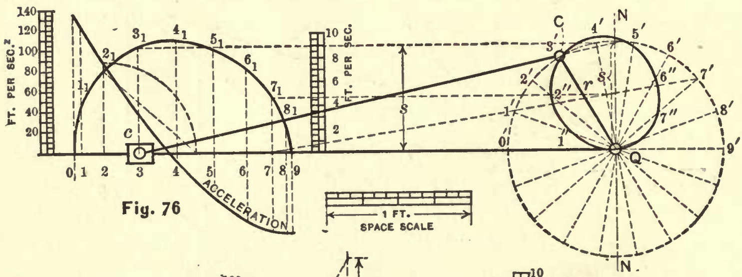

Fig. 76

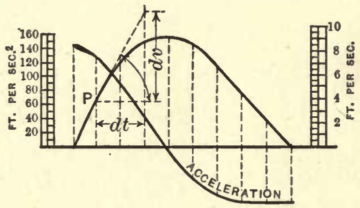

Fig. 76 (a)

the reciprocating engine in outline. C is the crank-pin, c is the crosshead-pin, Q is the centre of the shaft. The crosshead moves from 0 to 9 and back again to 0 during one complete rotation of the crank. The simultaneous positions of crosshead-pin and crank-pin are indicated respectively by 0, 1, 2, 3, etc., and 0′, 1′, 2′, 3′, etc. As shown in the preceding article, if the linear velocity of the crank-pin is represented by the length of the crank, r, the velocity of the crosshead for any phase is represented by the segment, s, of

the line N-N, which lies between Q and the line of the connecting rod, C-c. If the segment, s, is found for each of the crosshead positions from 0 to 9, the corresponding lengths of s may be erected as ordinates to 0–9 at the corresponding crosshead positions. A curve passing through the upper ends of these ordinates gives a velocity diagram of the point c with the path, 0–9, as a base. This diagram is called a *Velocity-Space Diagram.* If a sufficient number of ordinates have been determined the diagram gives quite accurately the velocity of c for intermediate positions.

Fig. 77 shows a method of constructing a velocity diagram upon

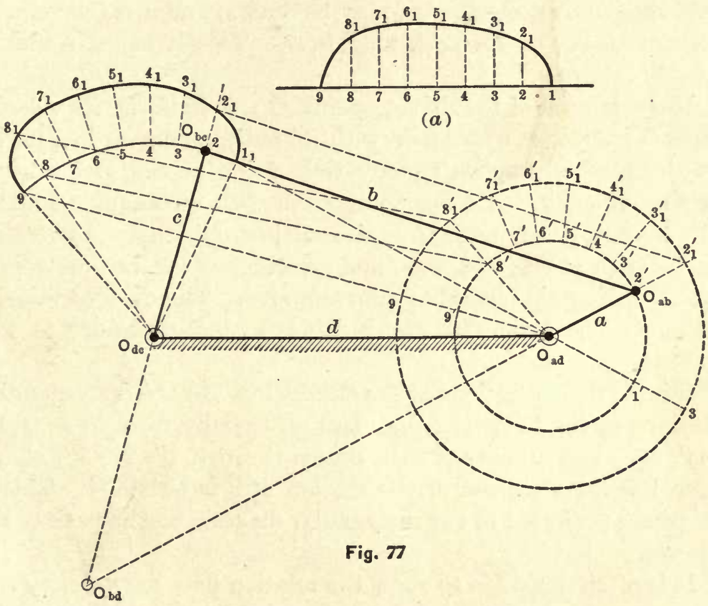

Fig. 77

a curved path as a base. The driving arm, or crank, a, imparts, by its rotation, a reciprocating motion to the arm c in the arc 1–9.

The point O_{bc} occupies the positions 1, 2, 3, etc., when the point O_{ab}, is at the corresponding points 1′, 2′, 3′, etc. If 2′–2_1′ is laid off equal to the linear velocity of O_{ab} upon the extension of

the line of a, and $2_1'$–2_1 is drawn parallel to b, the segment of the extension of c cut off by this parallel equals the linear velocity of the point O_{bc}. This is proven by reference to the instant centre of b and d, O_{bd}; for the linear velocity of O_{ab} relative to d is to the velocity of O_{bc} as O_{bd}–O_{ab} is to O_{bd}–O_{bc} (the linear velocities of two points in b relative to d are proportional to their radii from O_{bd}). But $2'$–$2_1'$ (the velocity of O_{ab}) is to 2–2_1 as O_{bd}–O_{ab} is to O_{bd}–O_{bc}, and therefore 2–2_1 is the velocity of O_{bc}. By a similar construction for other phases, the corresponding velocities of the point O_{bc} may be obtained. If these velocities of the driven point are laid off as radial ordinates at the corresponding points in its path, the curve 1_1–2_1–3_1, etc., may be drawn, and it is the velocity diagram of O_{bc} on its path as a base. This is called a *Radial Velocity Diagram.*

If the motion of the driving point, O_{ab}, is uniform, its velocity diagram is a circle concentric with its path, as drawn in Fig. 77, but the method applies equally well if the driving point has a variable velocity. A velocity diagram with rectangular co-ordinates may be constructed from the one just determined by rectifying the path of O_{bc}, 1–2, etc., and erecting, at the various points, parallel ordinates of lengths found as above. This derived velocity diagram is shown in Fig. 77*a*, but it is seldom necessary to construct it.

If on various positions of the crank (Fig. 76) the corresponding velocities of the follower, c, are laid off radially from Q, as Q–$1''$, Q–$2''$, etc., and a curve is then drawn through $1''$, $2''$, $3''$, etc., a *Polar Velocity Diagram* of the motion of c is obtained. This is sometimes preferred to the rectangular diagram on the path of the follower.

It is often desirable to show the relation between velocity and time. For this purpose a diagram may be constructed (Fig. 76*a*) in which ordinates represent velocity and abscissas represent time. This is called a *Velocity Time Diagram.*

In the illustrations of Arts. 39, 40, and 41, linkwork mechanisms have been taken, as the methods developed in these articles are especially useful in the treatment of this class; but the deductions are also applicable to other mechanisms.

42. Acceleration Diagrams.—In any velocity-space diagram the subnormal to the curve at any point is proportional to the corresponding acceleration. When different scales are used for velocity (ordinates) and for space (abscissas), as is usually the case, still another scale must be used for acceleration.

Let OPQ, Fig. 78, be any velocity-space curve in which 1″ of ordinate represents n times as many velocity units (ft. per sec.) as 1″ of abscissa represents space units (ft.).

Let PM, PT, and PN be the respective ordinate, tangent and

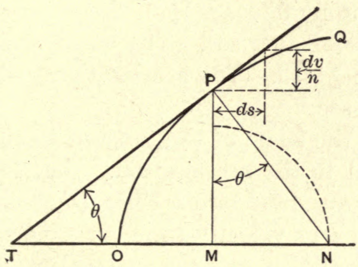

Fig. 78

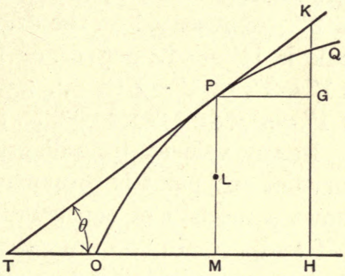

Fig. 78*a*

normal to the curve at any point, P, and let θ be the angle between the tangent, PT, and the base line, TN.

$v=\frac{ds}{dt}=nPM=$velocity represented by the ordinate PM, when the length PM is measured by the space scale;

$p=\frac{dv}{dt}=$corresponding acceleration;

$$\tan\theta=\frac{PM}{TM}=\frac{MN}{PM}=\frac{dv\div n}{ds}=\frac{1}{n}\frac{p}{v}\frac{dt}{dt};$$

$$\therefore \qquad p=MN\times n\times\frac{v}{PM}=n^2MN.$$

Hence the subnormal, MN, is proportional to the acceleration and may be used as an ordinate, at M, of an *Acceleration Space Diagram* (see also Fig. 76). When so used 1″ of ordinate represents n^2 times as many acceleration units (ft. per sec.^{2}) as 1″ of abscissa represents space units (ft.). Since 1″ of velocity

ordinate represents n times as many velocity units as 1″ of abscissa represents space units, 1″ of accleration ordinate represents $\frac{n^2}{n}=n$ times as many acceleration units as 1″ of velocity ordinate represents velocity units.

The engine mechanism of Fig. 76 is drawn to a scale of $\frac{1}{8}$ size.* The ordinates of the velocity curve represent the velocity of the cross-head to a scale on which the length of the crank on the drawing measures the linear velocity of the crank-pin. Taking this velocity as 9 ft. per sec. and the actual length of the crank as 9″ (represented on the drawing by $9'' \times \frac{1}{8} = 1\frac{1}{8}''$), the velocity scale is $1\frac{1}{8}'' = 9$ ft. per sec. or $1'' = 8$ ft. per sec. The space scale is $1'' = 8''$, or $1'' = \frac{2}{3}$ ft. $\therefore n = 8 \div \frac{2}{3} = 12$. The acceleration scale is $1'' = n^2 \times \frac{2}{3} = 144 \times \frac{2}{3} = 96$ ft. per sec.^{2}

In any velocity-time diagram the slope of the tangent to the curve at any point is proportional to the acceleration. Fig. 78*a* shows a method of constructing an *Acceleration-Time Diagram.* PM and QH are two ordinates of any velocity-time curve OPQ, at any convenient distance apart, TK is tangent to OPQ at P, and makes an angle, θ, with the base line, TH. QH is extended to cut TK at K, and PG is drawn parallel to TH. From M take $ML=KG$ as an ordinate of the acceleration curve, and determine other ordinates in the same way, the distance between the two ordinates used being equal to PG in each case.

In Fig. 78*a*, $p=\frac{dv}{dt}=\tan\theta=\frac{KG}{PG}$. Therefore KG measured in velocity units is the acceleration in corresponding acceleration units during an amount of time represented by the length of PG. When $PG=\frac{1}{m}$ sec., $p=mKG$. On the corresponding acceleration scale 1″ represents m times as many acceleration units (ft. per sec.^{2}) as 1″ on the velocity scale represents velocity units (ft. per sec.).

Using the same data as in the preceding example, i.e.: velocity crank-pin $=9$ ft. per sec., and length of crank $=9''=\frac{3}{4}$ foot, the crank rotates $\frac{9}{\frac{3}{4}\times 2\pi}=1.91$ times per sec. The time of 1 rev.

* The figure is reduced to about $\frac{1}{2}$ size in reproduction.

is $\frac{1}{1.91} = .52$ sec. This time is divided into eighteen equal parts in constructing the diagram in Figs. 76 and 76*a*, and two of these parts are used as the distance between ordinates in constructing the acceleration-time diagram. These two parts represent $\frac{.52 \times 2}{18}$ sec. $= .058$ sec. $m = \frac{1}{.058} = 17.3$. The acceleration scale is $1'' = 8 \times 17.3 = 138$ ft. per sec.^{2}

In the preceding discussion the foot-second system of units was used throughout. Any other system of units may be used in a similar manner and the corresponding scales determined by the same methods.

It may be noted that the acceleration is indeterminate, graphically, on the velocity-space diagram, where the curve crosses the axis of X. It can be found for several ordinates near that point and extended to the end position without much error. On the time-velocity diagram, however, it is wholly determinate. Both methods are open to the objection that considerable error is necessarily introduced in drawing tangents to curves which are not very well defined themselves.

If the weight of the moving body is known the force required to accelerate or retard it at any position can be found from the acceleration curve. If F be this force, W the weight of the body, and p the acceleration, $F = \frac{W}{g}p$. The acceleration can be read off from the acceleration scale at any point and the force corresponding may be found simply by multiplying the acceleration by $\frac{W}{g}$. Or a force scale may be constructed, as can readily be seen.

43. Centrodes and Axodes.—The instant centre for two bodies having plane motion may also be a permanent centre, in which case it remains a fixed point in both bodies; but in the general case the instant centre does not occupy the same position in either body for any two successive relative positions of these bodies, and the *locus of the instant centre* upon each of the bodies is called a *Centrode.* The instant centre is a point common to the two bodies for the instant, and therefore the two coincident points of the bodies which lie at this centre have for the instant no relative motion; but

any other two points (one in each of these bodies) do move relatively. The pair of centrodes traced on the bodies by the motions of the instant centre are tangent to each other at the instant centre; and as these contact points of the centrodes have no relative motion, the pair of centrodes roll on each other with a pure rolling action. Points in the pair of centrodes which previously coincided in the instant centre are now—in common with other points of the two bodies—rotating relatively about the present instant centre; and a similar remark applies to a pair of such points which may coincide in the instant centre at any succeeding phase.

Any plane motion between two bodies, whatever the mechanism adopted for producing this motion, is exactly equivalent to that resulting from the rolling upon each other of two members whose contact lines conform to the centrodes for this motion. The nature of this action may be made clearer by reference to Fig. 79. Let a and b be two points in the body A, moving rela-

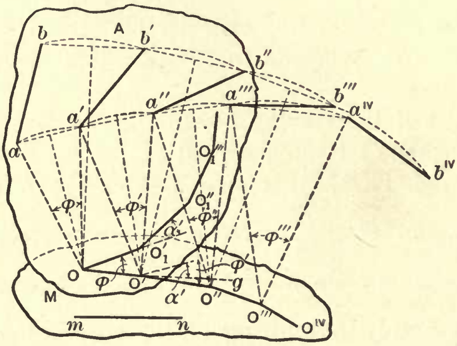

Fig. 79

tive to the fixed body M so as to occupy in succession the positions a-b, a'-b', a''-b'', etc. From the middle of a-a' and b-b' erect perpendiculars to these lines, intersecting in O; then the motion from a-b to a'-b' is equivalent to a rotation about the point O as a centre through the angle $aOa' = bOb' = \phi$. The motion a'-b' to a''-b'' is likewise equivalent to a rotation about O' through the angle ϕ', etc. O, O', etc., are temporary or shifting centres, and they may be connected by the polygon O-O'-O'', etc., which lies in the stationary body M. If the line O-O_1' be laid off on the moving body A of

a length equal to O-O' and making the angle ϕ with the latter, it is evident that when a-b moves to a'-b', O-O_1' will fall along O-O' and O_1' will coincide with O'.

From O' lay off an angle with O'-O'' equal to ϕ'; extend O-O' to g through this last angle; and let the angle $gO'O'' = \beta'$. This extension divides ϕ' into β' and α', and $\alpha' = \phi' - \beta'$. Extend O-O_1' to the right; from O_1' lay off the line O_1'-O_1'' equal to O'-O'' and making an angle with O-O_1' equal to α'. When a'-b' has moved to a''-b'', O_1'' will coincide with O''. By a continuation of this process the polygon O-O_1'-O_1''-O_1''', etc., is constructed on the face of the moving body A; and the given motion of A (a-b, a'-b', etc.) is equivalent to the broken rolling action of this polygon of A upon the polygon previously formed on M. If the positions of A (a-b, a'-b', etc.) are taken closer together, the corresponding positions of the temporary centres (O, O', etc.) become closer, and the polygons approximate more nearly to the centrodes for the given motion; and at the limit these polygons reduce to a pair of centrodes, and the temporary centres become true instant centres.

For other than a plane motion it has been seen (Art. 19) that the motion must be referred to a rotation about an axis instead of a centre. The *locus of the instant axis* is called an *Axode.*

Centrodes (or axodes) may be used in obtaining a motion which is too complex to get directly by the usual methods. Several desired positions of two points (a and b, Fig. 80) in a body A, relative to the points m–n of a body M, may be laid down, and the centrodes then derived by the process indicated above. If the two bodies A and M are attached to figures having these centrodes for contact surfaces, the simple rolling upon each other of these surfaces will produce the required motion. As will be shown in a later chapter, in treating the design of toothed gearing, it is possible to derive a pair of gears which will produce a motion identical with the rolling motion of these centrodes and free from any risk of slipping. It is mathematically possible to secure very complicated motions by the use of the principles given in this article; but there are many practical limitations to the applications of such a process.

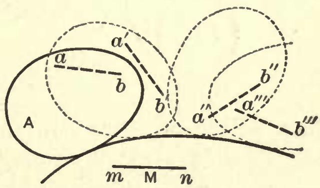

Fig. 80

CHAPTER III.

PURE ROLLING IN DIRECT-CONTACT MECHANISMS. FRICTIONAL GEARING.

44. Nature of Rolling Curves.—Since the condition of rolling action is that the contact point shall always lie in the line of centres, the contact radii must both coincide in direction with the line of centres to insure pure rolling, and as the contact radii lie in one straight line they make equal angles with the common tangent. In a pair of curves which roll upon each other (Figs. 81 or 82) let M and N be two points, one on each curve,

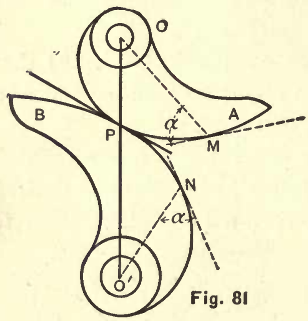

Fig. 81

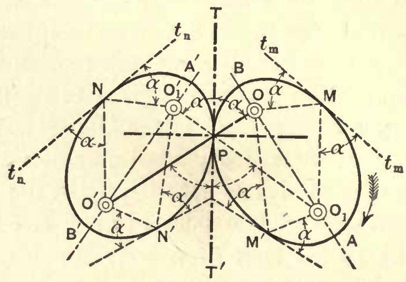

Fig. 82

that will come into contact when the radii OM and $O'N$ are in the line of centre. Then the radii OM and $O'N$ must make equal angles with the tangents to the curves at M and N, respectively; otherwise these radii could not lie in one straight line when the two tangents coincide at contact of M and N. Furthermore, the arcs PM and PN must be equal; and the sum of the radii OM and $O'N$ must equal the constant distance between centres O–O'; for if the first of these conditions is not satisfied, there must evi-

dently be some sliding action between the curves; if the second condition is not fulfilled, the two points M and N could not meet on the line of centres.

Besides pairs of circular arcs, in which the condition of pure rolling (but not that of positive driving) is met, there are many pairs of curves that satisfy the above conditions. Two of these forms will be treated in detail, and a general practical method will be given for deriving a curve which will roll with a given curve, the two centres being fixed.

45. Rolling Circles.—Figs. 64 and 65 show pairs of tangent circles which may roll upon each other, for the contact point always lies in the line of centres. The common normal passes through both centres in these cases so motion is not transmitted positively; but if we assume that there is no slipping between these curves the linear velocities of the points P_a and P_b are equal. If A makes n_1 revolutions, and B makes n_2 revolutions, per unit of time (calling the radius of $A = r_1$, and the radius of $B = r_2$), the linear velocity of $P_a = 2\pi r_1 n_1$, the linear velocity of $P_b = 2\pi r_2 n_2$ and, from the assumption of no sliding,

$$2\pi r_1 n_1 = 2\pi r_2 n_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The angular velocity of A is $\omega_1 = 2\pi n_1$, the angular of B is $\omega_2 = 2\pi n_2$, but from equation (1),

$$\frac{r_2}{r_1} = \frac{2\pi n_1}{2\pi n_2} = \frac{\omega_1}{\omega_2} = a\ constant; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

hence the angular velocities of A and B are inversely as their radii. This familiar relation corresponds with the relations given in Art. 34, where it was shown that in any case of rolling curves the angular velocity ratio is inversely as the lengths of the contact radii, or inversely as the perpendiculars from the fixed centres to the common tangent. This relation holds, whether the angular velocity ratio is constant as in the case of rolling circles, or otherwise. The general theorem of Art. 29, that the angular velocity ratio is inversely as

the perpendiculars from the fixed centres to the common normal, or inversely as the segments into which the line of centres is cut by the common normal is not applicable to the special case of tangent circles, for this normal coincides with the line of centres, and these ratios are indeterminate. Thus, the perpendiculars from O and O' upon NN' are both zero, and their ratio gives $\frac{\omega_1}{\omega_2} = \frac{0}{0}$.

The common point, P, of Figs. 64 and 65 may move either to the right or the left along the common tangent. It is evident from Fig. 64, in which the centres lie on opposite sides of the path of this point, that the rotations of A and B are opposite; if A has a right-hand, negative, or clockwise rotation, B has a left-handed, counter-clockwise, or positive rotation; or when the circles are in external contact their rotations are opposite. On the other hand, if the circles are in internal contact (one of them tangent to the concave side of the other, as in Fig. 65) the rotations are both in the same direction.

All the statements of this article apply to circular arcs rotating about their centres as well as to complete circles; except, of course, that unless the curves are full circles the action is limited, and must be reciprocating.

46. Rolling Ellipses.—Two equal ellipses, each rotating about one of its foci as a fixed centre, with a distance between centres equal to the common major axis, will roll upon each other without any sliding action.

In Fig. 82 two such ellipses are shown, with fixed centres at the foci O and O', and free foci at O_1 and O_1'. $OO' = AB = A'B'$.

It is a property of the ellipse that the lines drawn from any point (M) (Fig. 82) to the foci (O and O_1) make equal angles (α), with the tangent (t_m-t_m), and also that $OM + O_1M = AB$.

If N is a point similarly located in an equal ellipse, $O'N$ and and $O_1'N$ make an equal angle, α, with the tangent t_n-t_n. Now the two ellipses may be so placed together that M and N will coincide at the contact point, when the tangents t_m-t_m and t_n-t_n will become the common tangent, and OM and $O'N$ will lie in one

straight line, for they make equal angles with these tangents. If O and O' are made the fixed centres about which the ellipses rotate the contact point lies in the line of centres; hence the action is pure rolling. The distance $OO' = OM + O'N = AB$, as already stated. Also, $O_1O_1' = A'B' = AB = OO'$.

As M and N are *any* points similarly located in the two equal ellipses, the contact point will always be in the line of centres if the conditions as to these centres given at the beginning of this article be observed.

If there is no sliding between the two ellipses in acting through the angles POM' and $PO'N'$, respectively, (Fig. 82), the arcs PM' and PN' must be equal. This equality can be shown as follows:

$$OP + O_1P = AB = A'B' = O'P + O_1'P, \quad . \quad . \quad (1)$$

also $$OP + O'P = AB = A'B' = O_1P + O_1'P\,; \quad . \quad . \quad (2)$$

$$\therefore OP + O_1P = OP + O'P = O'P + O_1'P = O_1P + O_1'P. \quad . \quad (3)$$

From either the first and second, or the third and fourth members of (3) we get:

$$O_1P = O'P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

from which it is seen that the arcs PB' and PA are equal.

In a similar way it can be shown that $O_1M' = O'N_1'$, and that the arcs AM' and $B'N'$ are equal; therefore the arc $PM' = AP - AM'$ is equal to the arc $PN' = B'P - B'N'$. This demonstration is general and will apply to any pair of points which can meet as contact points. If the points P and M lie on opposite sides of AB, and P and N lie on opposite sides of $A'B'$, the values of PM and PN become $PB + BM$, and $PA' + A'N$, respectively, but the equality of the arcs is maintained.

The driving will be positive in the direction indicated, until the phase shown in Fig. 83 is reached, when the normal passes through both fixed centres, and the driver might continue to rotate without imparting further motion to the follower. To secure con-

tinuous driving for the half revolution succeeding this phase it must be provided for otherwise than by the simple contact of the two ellipses. It has been shown that the free foci O_1 and O_1', are always at a distance apart equal to the major axis, A-B, and these foci could therefore be connected by a link. This system of linkwork alone would transmit motion exactly equivalent to that of

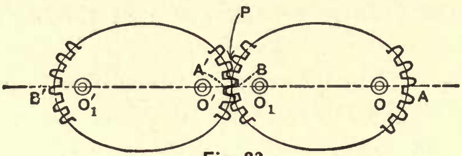

Fig. 83

the rolling ellipses; but in an actual mechanism the two pieces would have to be at the ends of the shafts between which motion is to be transmitted, or the link would interfere with the shafts.* Another obstacle to such a link connection, as a substitute for the rolling ellipses, is that at the phase shown in Fig. 83 (and at 180° from this position) the linkwork would reach a "dead-centre" position, when it would not be effective in transmitting motion.

Teeth may be placed at the ends of the elliptical members (as indicated in Fig. 83), which would engage near the dead-centre phases, and thus carry the follower past this critical position. If such teeth were placed around the entire halves of the ellipses which are in contact after the direct-contact driving ceases to be operative, the link could be omitted, and the necessity of placing the ellipses at the ends of the shafts thus avoided. Where the action is to continue through half a revolution, or more, such teeth are usually placed entirely around the peripheries of the ellipses, and the result is a pair of elliptical gears such as is shown in Fig. 84. The method of forming such teeth, to secure the exact equivalent of the rolling ellipses, will be discussed in a later chapter. With the transmission through such elliptical members

* It will be noted that the pair of rolling ellipses correspond to the centrodes of such a system of links as that just suggested.

as have just been discussed, the angular velocity ratio is inversely as the contact radii at any phase. If the driver has a uniform

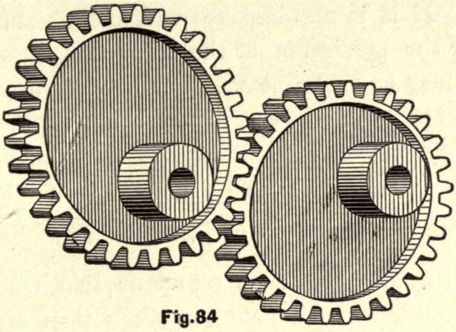

Fig. 84

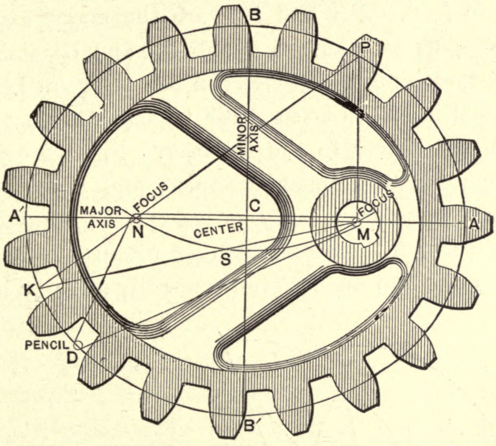

Fig. 84 A

angular velocity, the angular velocity of the follower is a maximum in the phase shown in Fig. 83, when $\frac{\omega_1}{\omega_2} = \frac{O'P}{OP} = \frac{OA}{OB}$. When the driver has made a half revolution from this position, the angular velocity of the follower is a minimum, and $\frac{\omega_1}{\omega_2} = \frac{OB}{OA}$. These

extreme ratios are reciprocals of each other. Of course the driver and follower both complete the half rotations from these two positions (where the contact radii coincide with the major axes) in equal times. If it is required to connect two shafts by rolling ellipses either the maximum or the minimum angular velocity of the follower may be taken at will, but one of these being determined the other is fixed—the driver being supposed to have a constant angular velocity.

Suppose it is required to construct a pair of rolling ellipses such that the maximum value of $\frac{\omega_1}{\omega_2} = \frac{2}{1}$. Divide the distance between centres OO' (Fig. 83) into such segments that $OP : O'P :: 2 : 1$. Lay off PA and PB' each equal to OO'; then lay off PO_1 and $B'O_1'$ equal to PO'. PA and PB' are the major axes of the required ellipses, whose foci are O and O_1, and O' and O_1', respectively; from these data the curves can be constructed.

Sectors of ellipses can be used for transmitting a reciprocating motion from the driver to the follower. In this case the angle through which one of the members turns, and both the maximum and minimum angular velocity ratios, can be assumed; but the angle through which the other member rotates is not then subject to control, for the two sectors are necessarily alike. Thus (Fig. 85)

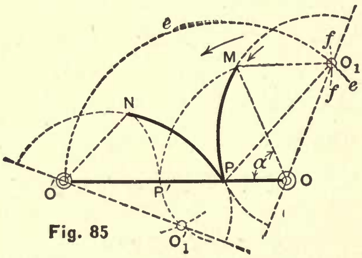

Fig. 85

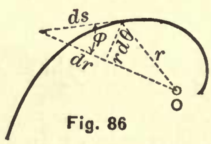

Fig. 86

the centres are at O and O', and it is required to construct a pair of elliptical sectors such that an angular motion, α, of the driver will transmit motion to the follower pivoted at O', and $\omega_1 \div \omega_2$ is to have for extreme values $O'P \div OP$, and $O'P' \div OP'$.

Draw a line from O making the angle α with OP, and on this line lay off $OM = OP'$. P and M are then points in the ellipse which rotates about one of its foci at O. The distance from P to the free focus of this ellipse, O_1, equals the major axis minus OP; or $O_1P = OO' - OP = O'P$. With this length as a radius and P as a centre draw an arc, *ee*. Also, the distance from O_1 to M, or $O_1M, = OO' - OM = O'P'$. With this length as a radius, and a centre at M, draw an arc *ff*. The intersection of the two arcs *ee* and *ff* is O_1. The foci being located and the major axis known, the ellipse can be drawn. The elliptical arc, PN, of the follower is equal to that of the driver.

The constructions just outlined apply either for actual rolling elliptical members, or for finding the "pitch curves" for toothed gears, or segmental gears.

Elliptical gears have been applied in many cases where a "quick-return" action is required, as to shaping-machines, in order to give a quick return motion to the tool with a slower stroke during the cutting. They have also been used to actuate the slide-valve in a steam-stamp used for crushing rock, where it is desirable to admit the steam above the piston throughout nearly the entire downward stroke in order to cause a more effective blow; while on the upward stroke economy demands that only sufficient steam be used to return the stamp-shaft.

47. Rolling Logarithmic Spirals.—One of the properties of the logarithmic spiral is that the tangent to the curve makes a constant angle with the radius vector at all points. Owing to this property, the curve is also called the *equiangular spiral.*

The polar equation of this curve is $\theta = \log_b r$, in which b is the base of the system of logarithms. The angle made with the tangent by the radii vectores is different for different values of b, but it is constant for any one system of logarithms.*

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* See Fig. 86, $\theta = \log_b r$. Let m = modulus of the system of logarithms, $\therefore\ d\theta = m\frac{dr}{r}$; but $\tan\phi = \frac{rd\theta}{dr} = \frac{rmdr}{r} \div dr = m$ = the modulus of the system of logarithms; $\therefore\ \phi = \tan^{-1} m =$ *a constant.*

If two similar logarithmic spirals are placed tangent to each other, as in Fig. 87 or 88, the tangents to the two coincident contact points lie in the same line; and as the angles made by these tangents with their radii vectores are equal, these radii lie in a straight line. This holds for all tangent positions of the curves; hence if the curves turn about fixed centres at their foci, the contact point always lies in the line of centres, thus meeting the requirement for pure rolling.

The sum of the contact radii if the foci are on opposite sides of the contact point, and their difference if the foci are on the same side of this point, is a constant and is equal to the distance between the fixed centres. Thus, in Fig. 87, $OP + O'P = OO'$;

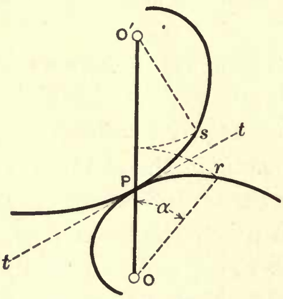

Fig. 87

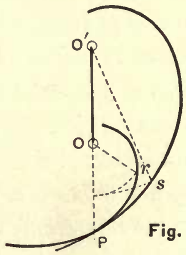

Fig. 88

and, if r and s are two points which may become coincident contact points, $Or + O's = OO'$. Also, in Fig. 88, $O'P - OP = OO'$; and, if r and s are two points which may become coincident contact points, $O's - Or = OO'$.

In Fig. 87:

$$OP + O'P = Or + O's, \quad \therefore Or - OP = O'P - O's. \quad . \quad (1)$$

In Fig. 88:

$$O'P - OP = O's - Or, \quad \therefore Or - OP = O's - O'P. \quad . \quad (2)$$

Equations (1) and (2) show that in either external or internal contact the difference between two contact radii of one of the

spirals equals the difference between the corresponding contact radii of the other spiral. It can be shown that any two arcs of similar logarithmic spirals are equal in length when the difference of the radii to the extremities of these arcs is the same. Hence in Figs. 87 and 88, $Pr = Ps$, as it should for pure rolling.*

A single pair of logarithmic spirals cannot transmit motion continuously in one direction, but they may be used for a reciprocating transmission with pure rolling. The angular motion of the driver and both extreme angular velocity ratios may be assumed, in which case the angle through which the follower moves can not be controlled. Thus, in Fig. 87, the driver may rotate about O through the angle $POr = \alpha$, and the angular velocity ratio varies from $O'P \div OP$ to $O's \div Or$. These conditions determine the points P and r in the spiral which has its focus at O. The focus O', the point P, and the *length* of a second radius vector, $O's = OO' - Or$, are also fixed for the second spiral; but as this must be similar to the first spiral, the angle $PO's$ cannot be assigned in advance. It is possible to fix the angles of motion of both driver and follower, but with these conditions only one angular velocity ratio can be taken arbitrarily.

48. General Case of Rolling Curves.—A general method will now be given for constructing a pair of curves which will roll upon each other in turning about two fixed centres. By this method the angular velocity ratios at the beginning and end of any angular motion of one member may be assigned; but the corresponding angular motion of the other member cannot be predetermined. Or, if one of the curves is prescribed, a curve can be found which will roll upon it. The method gives only approxi-

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* See Fig. 86. $\theta = \log_b r$; $m =$ modulus. $(ds)^2 = (rd\theta)^2 + (dr)^2$; but $d\theta = m\frac{dr}{r}$; $\therefore$ $(d\theta)^2 = \left(m\frac{dr}{r}\right)^2$, $\therefore$ $(ds)^2 = \left(rm\frac{dr}{r}\right)^2 + (dr)^2 = (m^2+1)(dr)^2$,

$\therefore$ $ds = \sqrt{(m^2+1)}dr$, $\therefore$ $s = \sqrt{(m^2+1)}\int_{r_2}^{r_1} dr = \sqrt{(m^2+1)}(r_1 - r_2)$; hence the length of the arc s included between two radii vectores of the same difference in length is constant.

mate results inasmuch as it does not absolutely insure theoretically perfect rolling between the points located; but the approximation can be carried to any required limit by locating a sufficient number of points.

Suppose the distance between the fixed centres, O and O', Fig. 89, to be given, and that it is required to construct a pair of rolling curves such that the angular velocity ratio of B to A shall be $OP \div O'P$, $OP_1 \div O'P_1$, $OP_2 \div O'P_2$, $OP_3 \div O'P_3$, etc., when the lines PO, m_1O, m_2O, m_3O, etc. respectively, lie in the line of centres; these last lines being drawn to correspond with required angular motions of A.

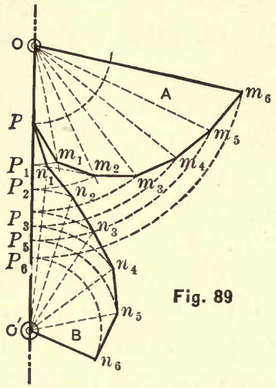

Fig. 89

The first pair of radii are OP for A, and $O'P$ for B. With O as a centre and OP_1 as a radius, describe the arc P_1m_1, cutting the line m_1O, then draw a line from P to m_1. With O' as a centre and $O'P_1$ as a radius, draw the arc P_1n_1; now take a radius equal to Pm_1, with P as a centre, and cut the arc P_1n_1 at n_1; and connect this point n_1 with P_1. It is evident that m_1 and n_1 can meet in the line of centres when A has turned through the angle m_1OP and B has turned through the angle $n_1O'P_1$. Next draw an arc through P_2, from centre O, cutting the line m_2O in m_2, and connect m_1 and m_2. Also draw the arc P_2n_2 with O' as a centre and $O'P_2$ as a radius; now with a radius equal to $m_1 m_2$, and with n_1 as a centre, cut this last arc at n_2; then draw the line n_1n_2. Proceed in a similar way with the points P_3, P_4, etc., locating the points m_3, m_4, etc., of A; and n_3, n_4, etc., of B. It will be seen that the polygons P–m_1–m_2 . . . m_6, and P–n_1–n_2 . . . n_6 may act together with a rough rolling action, and that two curves can be passed through P–m_1–m_2 . . . m_6, and P–n_1–n_2 . . . n_6, the action of which will closely approximate pure rolling if the points located are sufficiently close together; that is, if the arcs approximate the chords. Evidently,

if the outline of A had been given, the curve of B could have been derived by laying off the lengths Om_1, Om_2, etc., from O upon OO', and then proceeding as before in the location of the points of the outline B.

Fig. 90 shows the derivation of a curve B to roll upon the straight line which rotates about O as a centre and constitutes the acting line of A. The construction will be obvious from the preceding explanation in connection with Fig. 89.

This method cannot usually be applied where complete rotation of both of the members is required; for, as appears from the constructions given, the angular motion of the follower for a given

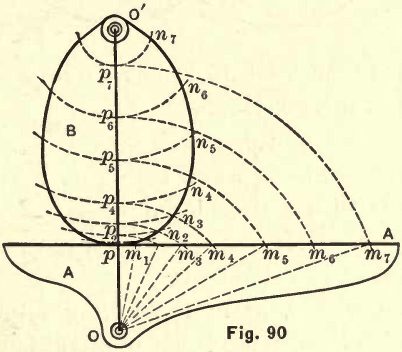

Fig. 90

motion of the driver cannot be controlled; hence it is not certain, in the general case, that a complete rotation of one member will correspond to a complete rotation of the other. But with continuous action in one direction, when one member has turned through 360° the other must have turned through an angle of 360°, or else some exact multiple or exact divisor of 360°. This requirement does not apply to rolling circles, but it holds for all other pairs of rolling curves.

49. Lobed Wheels.—It has been seen that a pair of equal ellipses can rotate continuously with rolling contact, and that the angular velocity ratio passes through one maximum and one minimum value for each revolution. It is sometimes desirable to have several maxima and minima values of this ratio to a single revolution, and

a class of rolling mechanisms called *Lobed Wheels* may then be used.

Fig. 91 shows a pair of these wheels, each having three lobes. The outlines are all logarithmic spirals.

If it be desired to have an unequal number of lobes on the two

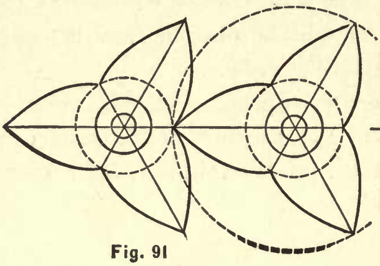

Fig. 91

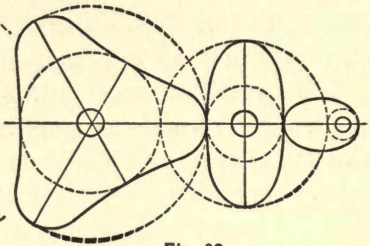

Fig. 92

wheels these spirals cannot be used; but curves which are derived from ellipses permit this condition.

Fig. 92 shows a set of three such wheels in series which roll perfectly; there is a one-lobed wheel acting on a two-lobed wheel, and this latter rolls with a three-lobed wheel. These figures are drawn from MacCord's Kinematics, to which the reader is referred for a full treatment of Lobed Wheels.

In all of these wheels, as in the rolling ellipses, there are periods during which the driving is not positive; but these outlines can be used as the pitch curves for toothed wheels, and teeth can be formed upon these curves which will transmit a positive motion exactly equivalent to that of the pure rolling of such curves. In these derived toothed wheels there is sliding between the teeth themselves, but no sliding (if the teeth are properly formed) between the pitch lines.

50. Rolling Surfaces.—In the preceding pages plane curves which roll upon each other while rotating about fixed centres have been considered. It was shown in Art. 10 that the plane motion of any body can be represented completely by the motion of a plane figure; thus these plane rolling curves may represent corresponding bodies which rotate about axes through the fixed centres.

and perpendicular to the plane of motion. When two or more such bodies can be represented by figures lying in the same plane, it is evident that the axes of all of these bodies must be parallel. The actual contact surfaces of such bodies are generated by a line which travels along the curved outline, always remaining parallel to the axes; hence these surfaces are cylindrical. The actual bodies corresponding to Figs. 64 and 65 are figures of revolution or right cylinders (see Fig. 93); while the bodies corresponding to Figs. 82 to 90 are cylinders only in the general sense. Certain other forms may roll together in rotating about fixed axes which are not parallel, when the motion of each member about its axis is still plane, but the planes of motion of the different members do not coincide. If the two axes intersect, tangent cones, or frusta (as in Fig. 95), having a common contact element and a common apex at the intersection of the axes, may act together with pure rolling. These cones are not necessarily right cones, but the use of cones of other than circular transverse sections is so rare that only right cones will be treated in this work.

If the two axes are not in one plane (*i.e.*, if they are neither parallel nor intersecting) they may still be connected by two members which will roll upon each other, with contact along a common rectilinear element. Fig. 101 shows the general form of a pair of such members; they are called *Hyperboloids of Revolution*. The general method of generating these latter figures and the nature of the action will form the subject of a later article, in which it will be shown that there is, in a sense, a certain departure from pure rolling in the action; however, this does not prohibit the use of these forms as pitch surfaces for toothed gears, owing to the peculiar character of the sliding component.

51. Rolling Cylinders.—In rolling right cylinders the angular velocities are inversely as the radii; or $\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$. Let d be the distance between the fixed axes. In external contact, $r_1 + r_2 = d$; and in internal contact $r_1 - r_2 = d$, (in this expression r_1 is taken as the radius of the larger cylinder, inside of which the smaller one

rolls). It is frequently required to find the diameters or radii of tangent cylinders which will connect two shafts and transmit motion (when there is no slipping) with a given angular velocity ratio. This ratio is the same as the ratio of the revolutions made in a given time by the two cylinders, and in practical problems it is usually stated in these terms. Thus, one shaft is to make n_1 revolutions imparting n_2 revolutions to the other shaft, per unit of time; then $\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{r_2}{r_1}$. In many cases the required radii, r_1 and r_2, can be found by inspection, or by mental calculation; but it may be convenient to use the following expressions if n_1 and n_2 are high numbers with no common divisor.

For Cylinders in External Contact: $r_1 + r_2 = d, \therefore r_1 = d - r_2$, and $r_2 = d - r_1$.

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}, \quad \therefore r_1 = r_2 \frac{\omega_2}{\omega_1} = (d - r_1)\frac{\omega_2}{\omega_1};$$

or $$r_1\left(1 + \frac{\omega_2}{\omega_1}\right) = d\frac{\omega_2}{\omega_1}; \quad \therefore r_1 = \frac{\omega_2}{\omega_1 + \omega_2} d; \quad r_1 = \frac{n_2}{n_1 + n_2} d. \quad (1)$$

Similarly: $$r_2 = r_1 \frac{\omega_1}{\omega_2} = (d - r_2)\frac{\omega_1}{\omega_2};$$

or $$r_2\left(1 + \frac{\omega_1}{\omega_2}\right) = d\frac{\omega_1}{\omega_2}; \quad \therefore r_2 = \frac{\omega_1}{\omega_1 + \omega_2} d; \quad r_2 = \frac{n_1}{n_1 + n_2} d. \quad . \quad (2)$$

For Cylinders in Internal Contact: $r_1 - r_2 = d$ (r_1 being the radius of the large cylinder). $\therefore r_1 = d + r_2$, and $r_2 = r_1 - d$.

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}, \quad \therefore r_1 = r_2 \frac{\omega_2}{\omega_1} = (r_1 - d)\frac{\omega_2}{\omega_1};$$

or $$r_1\left(\frac{\omega_2}{\omega_1} - 1\right) = d\frac{\omega_2}{\omega_1}; \quad \therefore r_1 = \frac{\omega_2}{\omega_2 - \omega_1} d; \quad r_1 = \frac{n_2}{n_2 - n_1} d. \quad . \quad (3)$$

Similarly: $$r_2 = r_1 \frac{\omega_1}{\omega_2} = (d + r_2) \frac{\omega_1}{\omega_2};$$

or

or $$r_2\left(1 - \frac{\omega_1}{\omega_2}\right) = d \frac{\omega_1}{\omega_2}; \quad \therefore r_2 = \frac{\omega_1}{\omega_2 - \omega_1} d; \; r_2 = \frac{n_1}{n_2 - n_1} d. \quad . \; (4)$$

The directions of the rotations of the two members are opposite when they are in external contact, and the same when one is tangent to the concave surface of the other, as previously pointed out.

52. Rolling Right Cones.—Two right cylinders, combined with two right cones, are shown in Fig. 94. Each cylinder has one base in common with that of one of the cones, hence the axis of this cylinder and cone must coincide. The bases of the two cones (and of the corresponding cylinders) need not be equal, but the

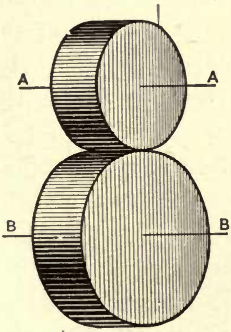

Fig. 93

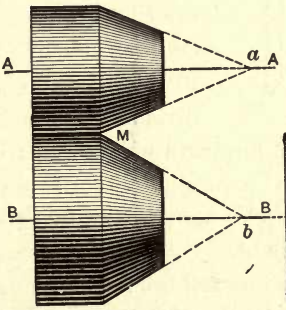

Fig. 94

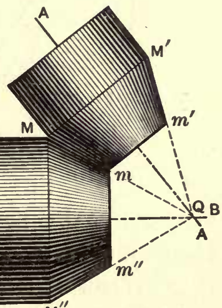

Fig. 95

slant height of both cones is the same. The bases of the two cones have a common tangent, in their plane (perpendicular to the paper), passing through M. Now imagine the two axes, AA and BB, to rotate in their common plane, about this tangent to the bases through M as an axis (or hinge), till the apex a meets the apex b at Q, as in Fig. 95; when the two cones become tangent along the element QM. It will be seen that the two base circles

still have a common tangent through M and they can roll upon each other in the new position, the two contact points having equal velocities along their common tangent, as in the original position. Any other corresponding transverse sections of the cones, equidistant from Q along the elements, as m-m' and m-m'' will also roll together; or the two cones roll upon each other in a similar manner to the original rolling of the cylinders.

If it is required to connect two given intersecting shafts by rolling cones, so that their rotations per unit of time shall be in the ratio of n_1 to n_2, it is only necessary to construct two tangent right cones with these shafts for axes, and with a common contact element lying in such a position between the axes that any pair of transverse sections which roll together shall have radii in the inverse ratio of the required angular motions. If A-A' and B-B', Fig. 96, are the given axes, the position of the contact element may be found by laying off from Q, on these shafts, the distances Qa and Qb, directly proportional to the required numbers of rotations of these shafts ; thus $Qa : Qb :: n_1 : n_2$. On Qa and Qb form a parallelogram, and the diagonal of this parallelogram, Qc, or its extension, is the required common contact element.

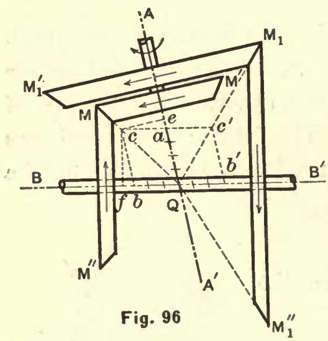

Fig. 96

This can be proved as follows : from c drop perpendiculars ce and cf upon the axes A-A' and B-B'; the angle $cbf = eac = \alpha$ (sides parallel); $ce = ca \sin \alpha$, $cf = cb \sin \alpha$.

$\therefore ce : cf :: ca : cb :: MM' : MM''$; hence the cones with MM' and MM'' as the diameters of the bases, will roll together with the required angular velocity. The frusta used for this transmission may be taken from any part of the two cones, giving bases greater or less than those indicated, if more convenient.

The parallelogram might have been drawn in any of the four

angles made by the intersection of A-A' and B-B'; thus if the angle $B'QA$ had been selected, the diagonal Qc' would have been located for the contact element, and two such frusta as those shown with M_1M_1' and M_1M_1'' as bases would give the required angular velocity ratio. Either of the other two angles, $A'QB'$ or $A'QB$, might have been taken if desired. It will be noticed that the cones first found are not similar to those obtained in the second construction; but the pairs constructed in both of the acute angles are similar, as are the pairs in both of the obtuse angles.

If the driving-shaft A-A' rotates as indicated by the arrows, it will be seen that the first construction (in the acute angle) imparts rotation to B-B' in one direction; while the second construction (in the obtuse angle) causes B-B' to rotate in the opposite direction. The choice of angle for the location of the contact element is governed by the required directions of the rotations, and the locations of the actual shafts. It is evident that one of the ma-

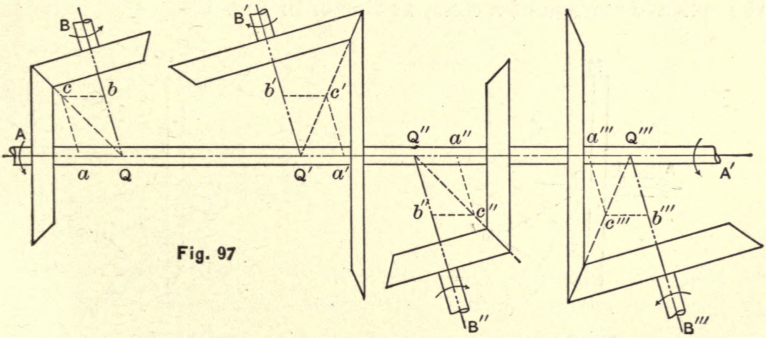

Fig. 97

terial shafts, but not both of them, can pass through Q. Fig. 97 shows a shaft A-A', from which four shafts (making equal angles with A-A') are driven. One of the followers on either side of A-A' is rotated in one direction; while the other followers (one on each side of the driver) rotate in the opposite direction.

It may happen, as in Fig. 98, that one wheel cuts through the

axis of the other wheel. The shaft can then be led off only in the direction indicated by the full lines; for if it were to be carried through Q, in the direction of the dotted lines, the shaft and wheel would interfere. This condition can only occur when the contact radius is located in the obtuse angles. The acute-angle construction is to be preferred as avoiding this difficulty in all cases, and also because it gives smaller wheels; but there are conditions as to location of shafts and required directional relation of rotation which may make the other construction desirable or necessary. The conditions of the problem may be such that the contact element is perpendicular to one axis, when the cone on this axis is of the special form (a flat disk) shown in Fig. 99. With somewhat different conditions, one of the rolling surfaces may be the concave surface of a cone, as shown in Fig. 100.

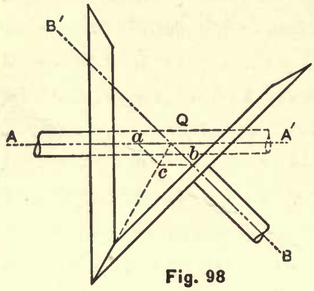

Fig. 98

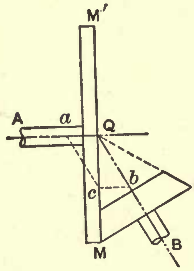

Fig. 99

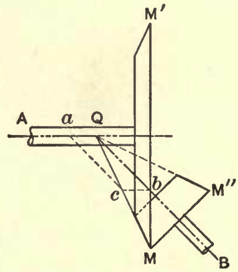

Fig. 100

In a great majority of the cases requiring the construction of rolling cones on intersecting axes, these axes are at right angles to each other. With this condition the pairs of cones formed in any of the four angles (for a given angular velocity ratio) have similar inclinations. The location of the contact radius in one of these angles, and the selection of the particular angle in which it lies, are determined by the general relations previously treated in this article.

53. Rolling Hyperboloids.—If one right line revolves about another right line not in the same plane, and all points in these lines remain at constant distances apart, the revolving line generates a surface called the *hyperboloid of revolution.* This is a warped surface, the elements of which are straight lines corresponding to the successive positions of the generating line. A meridian plane through this figure cuts the surface in an hyperbola, and it is evident that this hyperbola would generate a sur-

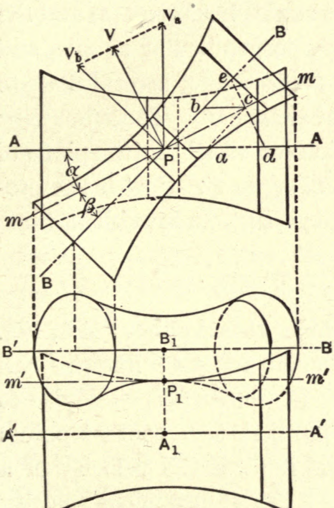

Fig. 101

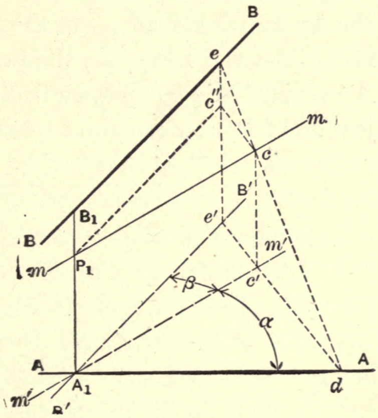

Fig. 101*a*

face, in revolving about the axis, identical with that generated by the straight line; hence the name given to these figures. Fig. 101 represents a pair of these hyperboloids of revolution tangent to each other along a common element *mm.* If the axes are fixed in the positions corresponding to this tangency, it is evident that the two surfaces will remain tangent as the two figures rotate about their axes; for each is symmetrical about its axis

Hyperboloids of revolution can be placed tangent along an element only when the radii of the "gorge circles" are proportional to the tangents of the angles between the contact element

and the respective axes; i.e., when $P_1A_1 : P_1B_1 :: \tan\alpha : \tan\beta$. This is shown in Fig. 101*a*, where AA and BB are the two axes and mm is the common element. A_1B_1 is perpendicular to both axes, and P_1A_1 and P_1B_1 are the respective radii of the gorge circles. These radii are normal to the hyperboloids and intersect the common element, mm, which is therefore perpendicular to A_1B_1 at P_1, and parallel to a plane through AA perpendicular to A_1B_1. $B'B'$ and $m'm'$ are the projections of BB and mm on this plane, and α and β are equal to the angles between mm and the respective axes. The lines cd and ce, perpendicular to mm, and intersecting AA and BB at d and e, respectively, are normals to the hyperboloids at c, on the line of tangency. Therefore they lie in one right line, de, the projection of which on the plane of AA and $B'B'$ is de', perpendicular to $m'm'$ at c'. P_1c'' is the projection of P_1c on the plane of BB and A_1B_1. It is evident that

$$\frac{\tan\alpha}{\tan\beta} = \frac{c'd}{c'e'} = \frac{cd}{ce} = \frac{cc'}{c''e} = \frac{P_1A_1}{P_1B_1}.$$

All points in the hyperboloid which rotates about AA, Fig. 101, must move in planes perpendicular to AA; likewise, all points in the other hyperboloid move in planes perpendicular to BB, and as the two axes are not parallel, two contact points can not have identical motions. Thus if V_a is the velocity of a contact point in the former figure, V_b is the simultaneous velocity of the corresponding point in the latter figure when they roll together. These two velocities must have equal components perpendicular to the contact element, but their components along this common line will not coincide. This is the characteristic of the action of these bodies referred to in Art. 50, and, as stated there, it does not affect the angular velocity ratio of the two members, for this relative sliding along the common element can not transmit motion, nor can it affect the component of V_a and V_b perpendicular to the common element.

The angular velocities of the hyperboloids (Fig. 101) when they roll together are, repectively, $\omega_1 = V_a \div P_1A_1$ and $\omega_2 = V_b \div$

P_1B_1. $V_a = V \div \cos\alpha$, and $V_b = V \div \cos\beta$. It has been shown that $P_1A_1 : P_1B_1 :: \tan\alpha : \tan\beta$. Therefore

$$\frac{\omega_1}{\omega_2} = \frac{V_a}{P_1A_1} \cdot \frac{P_1B_1}{V_b} = \frac{V\tan\beta\cos\beta}{V\tan\alpha\cos\alpha} = \frac{\sin\beta}{\sin\alpha}.$$

To construct a pair of rolling hyperboloids to transmit motion between two shafts with a given angular velocity ratio:—project these shafts on a plane parallel to both of them, Fig. 101: lay off Pa and Pb on AA and BB proportional to the required revolutions; construct the parallelogram P-a-c-b, and draw Pc; this locates the projection of the contact element. At any point c on Pc erect a perpendicular, cutting AA and BB in d and e respectively. Divide the perpendicular distance (A_1B_1) between AA and BB, at P_1, in the ratio of the segments cd and ce; then P_1A_1 and P_1B_1 will be the radii of the gorge circles of the required hyperboloids.

54. Frictional Gearing.—It has been shown that two axes, whether parallel, intersecting, or neither parallel nor intersecting, may be provided with contact members the surfaces of which will roll upon each other. In many mechanisms it is necessary to maintain, exactly, a prescribed relation between the motions of the members throughout the entire cycle of operations. In other instances this is not essential, a reasonable departure from the precise relative motions contemplated being permissible. Thus in cutting a screw-thread in a lathe, it is essential that the relation between the rotation of the spindle and the translation of the tool shall be strictly constant, and the positive mechanism (gears and the lead screw) insure this uniformity of action. But in plane turning the feed may vary somewhat without serious results, and the belt-driven rod-feed, depending upon friction, is often used, thus saving unnecessary wear of the screw. It sometimes happens, as in machinery subject to severe shock, that a positive transmission is not desired; and in many cases this is not an absolute necessity. When a limited variation of the motion transmitted may be permitted, and the two shafts to be connected are at a considerable distance apart, belting or rope transmission is most often employed. Occasionally, because the distance between the shafts is too small to employ these methods of transmission

advantageously, or for other reasons, the substitution of contact members rolling upon each other is convenient. In all such transmissions having circular transverse sections the action is purely frictional throughout the revolution, and these mechanisms are classed as *Frictional Gearing.*

If the sections are non-circular the action may still be pure rolling, as shown in the preceding chapter; but the driving cannot be positive during the entire rotation; for a critical phase is reached at which the action is only frictional, and beyond this phase driving does not occur, even by friction, unless other expedients (as teeth) are introduced (see Fig. 83). It is evident, then, that frictional gears must have circular transverse sections in order to transmit continuous rotation.

The force that can be transmitted through frictional gearing depends upon the physical character of the surfaces in contact and on the normal pressure between the two surfaces. Some slipping or "creeping" almost inevitably occurs; its magnitude depending upon the character of the surfaces, the normal pressure between them and the resistance to be overcome.

In certain applications this liability to slip is desirable rather than otherwise. For example, in hoisting, where it is not essential that the load raised shall move through precisely the same distance for each increment of motion of the driver. If any obstruction to motion of the load be met, the slip prevents the sudden strain (shock), that would be thrown upon the entire train of mechanism if this elasticity (using the word in a somewhat popular sense) were absent. If a car, or "skip," in being hoisted from a mine leaves the track, meets an obstruction, or is overwound, the yielding through the slipping of friction gears (or of belts) lessens the danger of breakage over that encountered with a positive connection. Furthermore, these friction mechanisms are much simpler in design and construction, and quieter in running than toothed gears; and, owing to such considerations, the employment of frictional gears, or "*frictions,*" as they are frequently called for brevity, is not uncommon, under proper circumstances.

Frictional gearing is important in itself, and the study of it also affords a good basis for investigation of toothed gearing.

Kinematically, any of the figures of revolution which will roll together, as pairs of right cylinders, right cones, or hyperboloids of revolution, might be used as friction gears; but, practically, rolling cylinders (Fig. 93), and the disk and plate ("brush-wheel") (Fig. 102), are by far the most common as the basis of such gearing. Rolling cones are also used, but less frequently.

Two cylinders (Figs. 64, 65, and 93) may be used to transmit motion and energy, up to the limits fixed by the friction at the contact element. Supposing no slip to occur, any two contact points have the same linear velocity, and the angular velocities of the two members, A and B, are inversely as their radii.

If it is required to impart to a shaft a given number of revolutions per unit of time, from a shaft of given rotative speed, the distance between centres being also determined; the required radii can be found by the expressions of Art. 51. For example, $d = 48''$, $n_1 = 210$ rev. per min.; $n_2 = 270$ rev. per min.

$$r_1 = \frac{n_2 d}{n_1 + n_2} = \frac{270 \times 48}{210 + 270} = 27''; \qquad \therefore\ r_2 = d - r_1 = 48 - 27 = 21''.$$

The solution of the kinematic part of this problem is extremely simple.

55. Grooved Frictions.—The consideration of the force that can be transmitted by friction-gears involves the normal pressure and the coefficient of friction between the contact surfaces. This consideration often modifies the forms of the members, without altering the kinematic action; and in many cases it may be advantageous to use certain derived forms, known as *grooved frictions* or "V" frictions, in place of the fundamental rolling cylinders. Fig. 103 shows a pair of these derived forms in contact. It will be seen that the original, or ideal, rolling cylinders are replaced by rolls with circumferential grooves, the sections of which (in planes passing through the axis) are triangular, or more usually, trapezoidal.

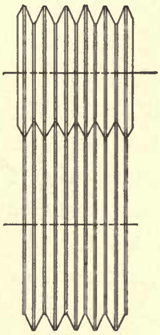

Fig. 103

The actual contact surfaces are frusta of cones of equal slant and on parallel axes.

In order to discuss the action of these grooved rolls, and to understand clearly their advantage over the simple rolling cylinders, it will be necessary to treat briefly the action of the forces involved in frictional transmission.

If two bodies are in contact, with a force F acting in the direction of their common normal, there is a resistance to the sliding of one body upon the other, and this resistance, called friction, is what makes frictional transmission possible. The resistance is a function of this normal pressure and of the physical character of the surfaces. If the surfaces are very smooth, the resistance under any normal pressure becomes comparatively small. If rough, the projecting particles of one member interlock with those of the other and the friction increases. As absolutely perfect surfaces are not attainable, absolute freedom from friction (absence of this resistance to sliding) is impossible; and the greater the departure from ideal perfection of surface (smoothness), the greater is the friction between any given pair of bodies. The friction varies inversely as the smoothness, and this varies both with the nature of the materials in contact and with the degree of "finish." In every case the friction is greater than zero; and the ratio of this resistance, f, to the normal force, F, is called the coefficient of friction, μ. This coefficient can only be derived from experiment, directly or indirectly.

Let the normal pressure between the surfaces of the two cylinders (Fig. 93) be represented by F. According to Newton's third law, action and reaction are equal and opposite; hence, the pressure of A towards B is met by an equal and opposite pressure of B towards A. These pressures can only be brought to bear upon the contact surfaces through the bearings of the wheels (neglecting weight), and the action and reaction at the bearings are equal; therefore a pressure F must be exerted by the bearings upon the axle supported by them. In other words, the pressure between the bearings and journals equals the pressure between the contact surfaces of the two wheels. As the bearings themselves, however per-

fectly formed and lubricated, are not frictionless, the normal force, F, necessary to transmit energy from A to B, involves a frictional action at the bearings, resulting in a wasteful resistance to be overcome, and also, incidentally, in wear of these parts. It is therefore desirable to reduce the pressure at the bearings as much as possible; but the friction at the contact surfaces must be sufficient for driving, and the normal pressure at these surfaces is one of the elements which determine this friction. It is in order, then, to investigate the relation between the bearing pressure and the normal pressure at the contact surfaces, and to see if the former can be reduced without undue sacrifice of the latter. With simple cylindrical rolls (Fig. 93) the total bearing pressure for each wheel equals the normal pressure, F, at the contact element. In the case of "V" frictions, however, the normal pressure between the contact surfaces may be much greater than the bearing pressure. This can be shown in connection with Fig. 104, in which the wedge of A is inserted in the corresponding groove of B. The common normals to the contact faces of A and B through the centres of the faces are Pn and Pn', and the normal forces between these faces may be taken as acting in the lines of these normals (such normal forces are really the resultants of systems of parallel forces, uniformly distributed over these faces). The force F, acting in the centre line of A and B as indicated, passes through P, and it can be resolved into components along Pn and Pn' by the parallelogram of forces. These components are represented by F_1 and F_1'. The effect of the initial force, F, is equivalent to the combined effect of its components, and it may be replaced by them; therefore, the effect of F is equivalent to the normal actions, F_1+F_1'.

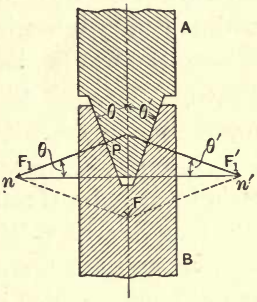

Fig. 104

$F_1 \sin\theta + F_1' \sin\theta' = F$; and $F_1 \cos\theta = F_1' \cos\theta'$. If $\theta=\theta'$ (the usual condition), $F_1 = F_1'$, and the total normal action, $F_1+F_1' = 2F_1 = F \div \sin\theta$. It is seen, from this last expression, that

the normal pressure increases, for a given value of F, as θ becomes smaller. When $\theta = 90°$, the total normal pressure equals F, as it should; for in this case the groove and wedge have disappeared and the contact surfaces are flat and perpendicular to the line of F. For any value of θ less than 90°, the total normal pressure is greater than F.

The action between the grooved faces of the "V" frictions is exactly like that of this wedge. The normal pressures between the sides of the acting ridges and the grooves correspond to the normal pressures in the wedge, and the initial force F equals the radial force exerted between the bearings and journals of the wheels. It follows from this discussion that any necessary normal pressure at the acting contact surfaces of the "V" frictions can be maintained by a bearing pressure less than the normal pressure. Hence, the friction loss in the bearings is decreased by the substitution of the "V" frictions for the fundamental rolling cylinders; or, to state it somewhat differently, for a given pressure at the bearings, a greater resistance can be overcome at the rim by "V" frictions than by cylindrical rolls. As there is a practical limit to the pressure that can be safely carried at the bearings, and as excess of bearing pressure means waste through friction, the importance of the wedge-like action in frictional transmission is apparent.

The angle between the sides of the grooves (2θ) is usually from 40° to 50°. Assuming 40° as this angle, $\theta = 20°$, and $\sin\theta = 0.342$. Then $2F_1 = \frac{F}{.342} = 2.93F$; or the total resulting normal pressure is nearly three times the force at the bearing, and the coefficient of friction and bearing pressure remaining the same, nearly three times as great a resistance can be overcome with these grooved rolls as with corresponding true cylindrical rolls.

Grooved frictions are frequently so mounted that the distance between the shafts can be changed somewhat. This makes it possible to throw the wheels out of gear, so that the follower can be stopped without checking the driver. It also permits controlling the bearing pressure, so that it need not be any greater than is required to prevent serious slipping at the driving surfaces.

This adjustment also affords a ready means of taking up the wear of the "V" or of the bearings, so that good contact (without which driving is impossible) is maintained.

When in gear (close contact) there is, as usually constructed, a small clearance at the bottoms of the grooves, as indicated in Fig. 104. If this were not provided, the edges of the rings might "bottom;" that is, the contact might be entirely or mainly at the bottoms of the grooves, instead of at the inclined sides. Such a condition would defeat the object of the grooves, and to render it impossible, even after considerable wear at the sides, this clearance is provided.

The depth of the grooves of either wheel is the difference between the radii to the tops and bottoms of the grooves or rings. This distance minus the clearance at the bottoms may be called the *working depth,* and the faces of the "Vs" above the clearance may be called the *working surfaces.* The nominal radius, or *pitch radius,* of a grooved friction may be taken as the mean radius of the working surface, and the hypothetical cylinder corresponding to this radius will then be the *pitch cylinder,* or pitch surface.

The angular velocity of two V friction wheels, when in full contact and working properly, may be taken, for most practical purposes, as that corresponding to the rolling together of the pitch cylinders, or, inversely, as the pitch radii. The relative sliding or creeping of the wheels along the common tangent to the pitch surfaces may usually be neglected in well-constructed frictions; for these wheels are only employed where some variations in the angular velocity ratio is admissible. Assuming that no sliding of this character takes place—that is, that the angular velocities of the two wheels are inversely as their pitch radii—there is nevertheless some relative motion between the two surfaces when they are in contact, causing a grinding action. The nature of this action may be seen in connection with Fig. 105, in which O and O' are the fixed centres of A and B, p is the contact point at the pitch circles, and s and t are two coincident points in the working surfaces, one on each side of p. The linear velocity of p, pv is assumed to be the

same for the coincident points of both wheels which lie at p. It is evident that all points in either wheel which lie outside of its pitch circle have linear velocities greater than pv, and all points of either wheel lying inside of its pitch circle have linear velocities less than pv; but those points of the working surface of one wheel which are inside of the pitch curve come in contact with points of the other wheel which are outside of its pitch circle; consequently, if the points at the pitch circles have equal linear velocities, all contact points not in these circles have different velocities, and there must be some relative motion or sliding between any pair of such points. Thus, in Fig. 105, the linear velocity of s, as a point in A, is sv'; and, as a point in B, the velocity is sv''; therefore the rate of sliding of these points equals $sv'' - sv'$. Similarly, the rate of sliding at t is $tv_1' - tv_1''$. An expression for the greatest sliding is derived below. Let R and r be the two pitch radii, N and n be the numbers of revolutions per unit of time of the corresponding wheels, and h the working depth of the grooves. Then the velocity of a point in the pitch circle of either wheel is $2\pi RN = 2\pi rn$ $\therefore$ $RN = rn$. An extreme outer point of the working surface of the first wheel has a radius $R + \frac{1}{2}h$, and it comes in contact with a point of the other wheel having a radius $r - \frac{1}{2}h$; hence the sliding at these points per unit of time equals

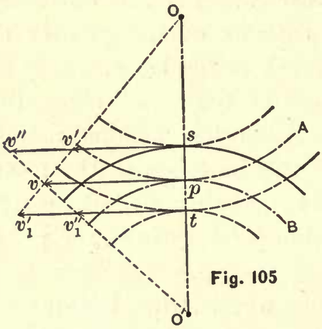

Fig. 105

$$S = 2\pi(R + \tfrac{1}{2}h)N - 2\pi(r - \tfrac{1}{2}h)n$$
$$= 2\pi[RN + \tfrac{1}{2}hN - rn + \tfrac{1}{2}hn] = \pi h(N + n), \text{ since } RN = rn.$$

By taking extreme contact points on the other side of the pitch circles, having radii $R - \frac{1}{2}h$ and $r + \frac{1}{2}h$, the same result can be reached by a similar process. The grinding action just noted tends to wear the working faces, even if no slipping occurs at the pitch circles. Such action does not take place in simple cylinder friction-rolls, but it cannot be avoided if the grooves have sensible depth.

The rate of this sliding action is directly proportional to h; therefore the working depth should be as small as practicable. This dimension is limited in practice, without sacrifice of the necessary total contact surface, by using several grooves, side by side, as indicated in Fig. 103, instead of fewer and deeper ones.

56. Brush-wheels.—Fig. 102 shows a mechanism sometimes used where it is desired to vary the angular velocity of a shaft which is driven by another shaft of constant angular velocity. Suppose the plate on the shaft AA to be the driver, and the disk, or "brush wheel," on BB to be the follower. A long keyway, or spline (or its equivalent), permits the disk to be placed at different positions along a line parallel to a diameter of the plate, as indicated by the dotted locations. The disk is imagined to be of no sensible thickness; hence it touches the plate at a single point,

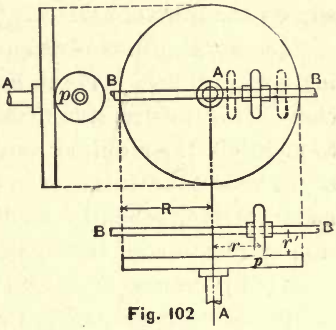

Fig. 102

p. This point, p, is at a distance from BB equal to the radius of the disk, r'; and at a distance from the axis AA equal to r, which may vary from zero to R (plus or minus). Assuming no slipping at p, the angular velocity ratio of AA to BB is $r' \div r$ (inversely as the radii). When the plane of the disk is in the axis AA, the velocity of p is zero, hence the follower is at rest. If the disk is carried beyond this position (to the opposite side of AA), the direction of the rotation of the follower is reversed. This mechanism is not well adapted for heavy forces; but is very convenient in many cases for light work, as in feed mechanism and for similar purposes requiring considerable change in the rotative speed of a follower, or reversal of direction of rotation. The disk must have sensible thickness in practical applications, and this gives rise to a grinding action somewhat similar to that mentioned with "V" frictions. If the disk is a cylinder, with contact between one of its elements and a radius of the plate, it is evident that all points in

this cylindrical element must have the same linear velocity (being points in a body at the same distance from the axis); while the corresponding contact points in the radius of the plate have different linear velocities (being at different distance from the axis of this plate). The disk should therefore be as thin as practicable, and its edge is sometimes rounded to approximate the point contact of the ideal disk. The plate should usually be the driver; for if this is not the case, when the disk is in contact with the centre of the plate, the latter is at rest, and the edge of the disk is compelled to slip on the contact surface.

The working disk is often made of leather, wood, or other yielding material, held between metal washers of slightly smaller diameter. This construction increases the adhesion, and makes it easier to maintain the required normal pressure at the contact point as slight wear takes place. In other friction mechanisms one of the members is frequently made of a non-metallic substance for a similar reason, and this member should usually be the driver; for if any slip occurs, by reason of the resistance being greater than the friction can overcome, the tendency is to wear off the edge of the rotating driver evenly, and to wear a depression, or notch, in the stationary follower. If the driver is made of the softer material the more irregular and objectionable wear of the follower is thereby reduced. In the brush-wheel mechanism it is not so easy to support the soft face on the driver (the plate), and there is not the same reason for doing so; because, even if the wear were all concentrated on this face, it would not be worn off evenly all over, for the follower only covers a small portion of its working surface in any position. When the follower (disk) is at the centre of the plate there is a tendency to wear a small flat place on the edge of the former. This may be avoided in many cases by cutting a slight depression at the centre of the plate, so that contact does not take place in this position of the disk.

57. Cone Friction.—Intersecting axes are sometimes connected by rolling conical friction wheels similar to the arrangements indicated in Figs. 96 to 100; but these are not so satisfactory as the

frictions on parallel axes, as it is more difficult to adjust the positions of the shafts to maintain the required normal pressure. If the force to be transmitted between intersecting axes is considerable, it may be better to use positive connections, as bevel-gears, to connect the intersecting shafts, and to introduce the friction element, if necessary, by means of a supplementary shaft parallel to one of these.

Two cones, as shown in Fig. 106, are sometimes used to connect parallel shafts, where changes in the angular velocity of the follower are required. These two cones are similar in inclination, and placed with the adjacent elements parallel, but not touching. An intermediate disk, C (or its equivalent), capable of being moved along the lengths of the cones, is in contact with both of them. Assuming no slipping at either contact, the linear velocity of the edge of this disk will be that of the part of the driver with which it is in contact, and this same linear velocity will be imparted to the follower; hence the linear velocities of the contact points of the two cones will be equal, and the angular velocities will be inversely as the contact radii of the cones at these points. If the disk is placed nearer the large base of the driver it acts on a smaller section of the follower, and the angular velocity of the latter is correspondingly increased.

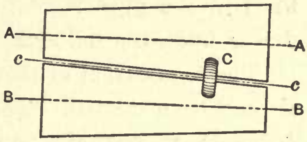

Fig. 106

This device, or modifications of it, is now on the market, for use as a countershaft. Provision is made for maintaining proper contact between the disk and the cones. In this case, as in that of the brush-wheel, the disk must have appreciable thickness; hence its contact element engages with points on the cones which must have somewhat different linear velocities, and a corresponding grinding action occurs. Similar remarks as to the means of reducing the practical effect of this action apply to both cases.

CHAPTER IV.

OUTLINES OF GEAR-TEETH. SYSTEMS OF TOOTH-GEARING.

58. Pitch Surfaces.—It has been shown that many pairs of bodies (as cylinders, cones, etc.) may transmit motion from one to the other with pure rolling, while these bodies rotate about axes fixed in the proper relative positions; but that the action of the driver upon the follower is not continuously positive.

The application of these rolling bodies as frictional gearing has already been treated. There are many cases, however, where it is desirable to secure a motion equivalent to one of these rolling actions, but where it is absolutely essential that no practical variation from this prescribed motion shall occur. This requirement is frequently met by using the surfaces of the appropriate rolling members as bases, and attaching interlocking teeth to them to prevent slipping. These rolling surfaces, when so used, are called pitch surfaces; and sections of them perpendicular to their axes are called pitch lines, or pitch curves.

Toothed gearing may he classified according to the pitch surfaces, relation of the axes, and character of the tooth elements as follows: *

	Class of Gearing.	Relative Position of Axes.	Pitch Surfaces.	Elements of Teeth.
1	Spur	Parallel	Cylinders	Rectilinear
2	Bevel	Intersecting	Cones	"
3	Skew	In different planes	Hyperboloids	"
4	Twisted	Any	Either	Helical
5	Screw	In different planes	Cylinders	"
6	Face	Any	None	Circular

* MacCord's Kinematics.

The action of these various classes will be treated in detail in later articles.

The two tangent circles of Fig. 107, representing rolling cylinders, may have their circumferences divided up into arcs of equal

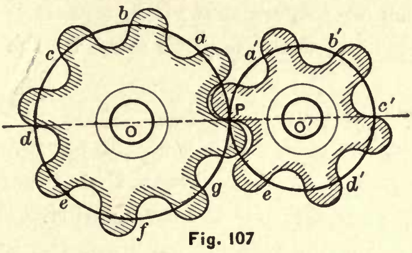

Fig. 107

length, $p = Pa = ab = bc$, etc., $= Pa' = a'b' = b'c'$, etc. This length of arc, p, must be a common divisor of both circumferences, and the numbers of divisions on the two circles are proportional to their circumferences, diameters, or radii. Let the radii be represented by r and r', and the numbers of divisions of the respective circles be called t and t'; then as

$$\frac{2\pi r}{p} = t, \quad \text{and} \quad \frac{2\pi r'}{p} = t'; \quad p = \frac{2\pi r}{t} = \frac{2\pi r'}{t'}.$$

As t and t' are directly proportional to r and r', it follows that the angular velocity ratio is inversely as the number of the divisions of the two circumferences. It is to be noticed that a and a', b and b', c and c', etc., are pairs of points which become coincident contact points as the circles roll together.

Now if we bisect the arcs Pa, ab, Pa', $a'b'$, etc., and place projections and corresponding notches on the alternate subdivisions, as indicated by the shaded outlines of Fig. 107, it will be seen that the wheels resemble, somewhat, the familiar toothed gears. The part of the tooth outside of the pitch circles is called the *addendum* or *point;* the portion inside of the pitch circle, between the spaces, is called the root. The acting surface of the point, or addendum, is called the *face,* and the acting surface of the root is called

the *flank*. By the formation of such teeth the pitch circles have lost their physical identity, but they are, nevertheless, important kinematically as the basis of the toothed wheels. The distances Pa, ab, Pa', etc., from any point on one tooth to the corresponding point of the next tooth of the same wheel, measured on the pitch curve, is called the *circumferential pitch, circular pitch*, or simply the *pitch*. It is evident that the pitch must be the same for both wheels.

If these wheels are "meshed" (that is, placed with a tooth of one in a space of the other, and with the pitch curves tangent), as shown in Fig. 107, it is apparent that the rotation of one of them will cause the other one to rotate, and that the transmission is now positive. As this rotation goes on, the successive pitch points of the teeth of the two wheels come into contact on the line of centres, and the *mean* angular velocity ratio for complete rotations, or for angular motions of the wheels measured by their pitch arcs, is identical with that due to the pure rolling of the pitch circles. This might be sufficient for some purposes; but we have, as yet, no assurance that this angular velocity ratio is strictly constant throughout the angular movements corresponding to the pitch angles. That is, the *mean* angular velocity ratio during such an angular motion agrees with that of the rolling circles; but at any phase intermediate between contact at two pitch points the angular velocity ratio may be either greater or less than this mean. It is imperative in many cases, and desirable for smoothness of action and quiet running in nearly all cases, that the angular velocity ratio be constant for all phases.

59. Conjugate Gear-teeth.—The condition of constant angular velocity ratio in direct contact is that the common normal to the acting faces, through the point of contact, shall always cut the line of centres in a fixed point; hence the desired constancy of this ratio in such wheels as those of Fig. 107 demands that the common normal shall always pass through the point marked P. If the teeth are of such form that this condition is met, the motion transmitted is exactly equivalent to the rolling of the pitch circles,

otherwise there is some departure from the required relative motion.

In general, the form of the teeth of one wheel may be taken quite arbitrarily, and an outline can be found for the teeth of the other wheel which will give the required angular velocity ratio at all phases; but this statement is subject to practical limitations. A pair of teeth which work together properly are called *conjugate teeth*.

A practical mechanical method of finding a conjugate tooth outline, when both pitch curves and the form of the tooth to be mated are known, will be explained before treating the formation of teeth geometrically. This method is applicable when it is required to construct a wheel to mesh with an existing gear, whether the teeth of the latter have lost their original form through wear or not; and whether the pitch curves are circles or not.

Cut out two segments of wood, *A* and *B* (Fig. 108), corresponding to the two pitch curves, and mount them on centres properly located. Upon the segment *A*, representing the existing gear, attach, in proper position, a sheet metal templet corresponding in form to one of its teeth, and have this slightly raised above the surface of the wooden segment by inserting a piece of thick paper or cardboard between them, so that a piece of drawing-paper attached to the segment *B* can pass under the templet.

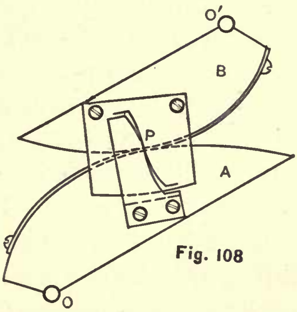

Fig. 108

Now roll the segments, without slipping, and trace the outline of the templet on the paper attached to *B* in several positions quite close together; a curve tangent to all of these tracings of the templet is the required tooth outline for *B*. A thin strip of metal between the edges of the two segments, one end of which is attached as indicated to each of the segments, will prevent slipping during the operation. It is evident that as *B* is rolled back and forth upon *A* the outline just derived on *B* will always be tangent to the

tooth of A, and if B is provided with a tooth of this form such a tooth in acting upon the given tooth of A will transmit motion identical with that due to the rolling of the pitch curves.

The method just explained is convenient for use in the shop, and it suggests a corresponding process for the drafting-room.

Draw the given tooth and its pitch curve upon a piece of heavy paper, and then draw the pitch curve of the other member upon tracing-paper, thin celluloid, or other transparent material. Place this last drawing above the other, with proper tangency of the pitch curves, and trace the outline of the given tooth upon the tracing-paper; roll the curves through a small arc, being careful to avoid slipping, and trace the tooth outline in its new position; repeat this operation until the entire arc of action of the teeth has been covered, and then draw on the tracing-paper a curve tangent to all of the tracings of the given tooth. This tangent curve is the required tooth outline.

From what has preceded, it will be seen that two cylinders may be provided with teeth such that the positive motion transmitted from one to the other will be identical with that of the two cylinders when rolling upon each other without sliding. This applies to cylinders other than those of circular cross-section; for the methods of finding a conjugate tooth, as given above, apply to any pair of rolling curves, such as rolling ellipses, logarithmic spirals, etc.

60. General Method of Describing Tooth Outlines.—The general method of describing gear-tooth outlines by means of an auxiliary rolling curve, or generator, will be developed in this article.

Suppose A and B (Fig. 109) to be any two rolling plane figures upon the outlines of which a pair of gear-teeth are to be described. As the pitch lines are rolling curves their point of contact is always on the line of centres. In the phase shown by the full lines, the angular velocity ratio of A to B is $O'P \div OP$; in the phase indicated by the broken lines, this ratio $O'P' \div OP'$; or for any phase of these rolling curves, the angular velocities of the members are inversely as the contact radii. If a pair of teeth give

a motion identical with that due to rolling of the pitch curves, it is evident that the common normal to the two teeth in contact must always pass through the point on the line of centres at which the pitch curves are tangent to each other; for these teeth are examples of direct contact members, in which the angular velocities are inversely as the segments into which the line of the normal cuts the line of centres.

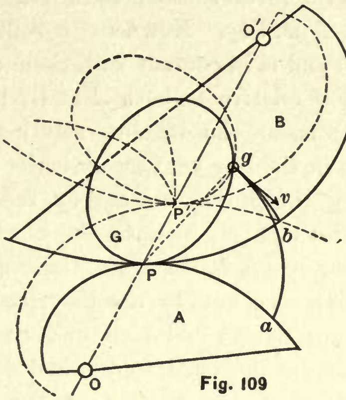

Fig. 109

If such a figure as G be rolled upon the convex side of the pitch curve of A, the point g of the figure G will trace the curve ga on the plane of A. Likewise, by the rolling of G on the concave pitch curve of B, the point g will generate the curve gb on the plane of B.

The curve G is the generating line of the teeth outlines, and it may be any line capable of rolling on the convex side of A and the concave side of B. The point, g, in this generating line is the describing point of the teeth. Now suppose the pitch curves and the generating line to be in the positions shown by the broken lines, with the generating point at P', the common point of tangency of the three lines. If A is turned to the right, as indicated by the phase shown in full lines, B will turn to the left in rolling upon it, and G can be rolled upon the pitch curves so that it remains tangent to both of them at their contact point in OO'. When the pitch lines have reached such a position as is shown by the full lines, G will lie in the position shown by the full line, and the original contact points of A, B, and G will be at a, b, and g, respectively. The arcs Pa, Pb, and Pg must be equal, as the action has been pure rolling. During this rotation the point g describes a curve upon the surface of A (this surface being supposed to rotate with A about O) such as ag, as noted above; g has, in a similar way,

generated a curve bg on the surface of B (rotating about O'), and, at the instant under consideration, g is the contact point common to ag and bg. Now as G is rolling upon the pitch curves of A and B, and is in contact with them at P, P must be the instant centre of G relative to both A and B; therefore the point g (a point in G) is, at the instant, rotating about P, and its motion must be in the line gv, perpendicular to gP. As the point g is generating the curves ag, and bg, the common tangent of these curves must coincide with the line of motion of g (gv), and gP, perpendicular to gv, is, therefore, the common normal to ag and bg. The curves ag and bg are described upon the surfaces of A and B, respectively; and it is evident that teeth upon these members, having the outlines ag and bg, will transmit a motion exactly corresponding to that of the rolling pitch lines; because their common normal passes through the point in the line of centres at which these rolling pitch curves are tangent to each other.

The reasoning of the foregoing discussion is perfectly general. It applies to any phase, if the condition that the three curves roll together with a common contact point is met at every instant of the action; hence the curves derived by this construction fully satisfy the kinematic requirements of tooth outlines.

The discussion immediately following will be confined to wheels having circles (or circular arcs) for pitch lines.

61. Usual Systems of Gearing.—There are a great many curves that can be used as generating lines for the outlines of gear-teeth, but only two are commonly used, viz., circles and right lines.

The curve traced by a point in a circle as it rolls upon the convex side of another circle is called an *epicycloid*; if it rolls upon the concave side of another circle, the curve traced is a *hypocycloid*; and if it rolls along a straight line a *cycloid* is described. When a right line rolls upon a circle any point in this line traces a curve called an *involute*.

The common systems of gearing in which the teeth are generated by circular or rectilinear describing lines are called, respectively, the *Epicycloidal System* and the *Involute System*.

62. Epicycloidal Gearing.—Fig. 110, A and B are two pitch circles, with centres at O and O', and tangent at the point P. The generator or describing circle, G, has its centre at o, on the line of centres OO'. If these circles all turn about their respective centres (rolling upon each other), the paths of these points, a, b, and g, which originally coincided at P, will be along the arcs Pa, Pb, and Pg. Since there is rolling contact Pa, Pb, and Pg are all of equal length. During this motion the point g will generate an epicycloid by rolling on the outside of A, and a hypocycloid by rolling on the inside of B. At any instant these two curves will be in contact at g, in the circumference of the describing circle. As the instant centre of the generator, relative to either of the pitch circles, is always at P, g moves perpendicular to Pg, and Pg is normal to both curves at their point of contact. This normal always passes through P, hence the angular velocity ratio is constant.

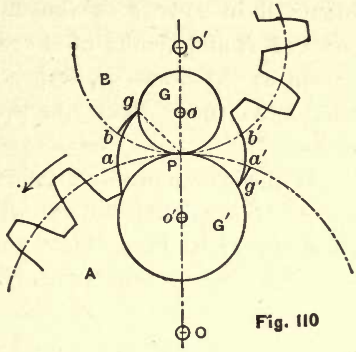

Fig. 110

The curves just discussed are suitable for the outlines of gear-teeth, and if the driver, A, has teeth with epicycloidal *faces*, and the follower, B, has teeth with hypocycloidal *flanks*, generated by the same circle, G, the action would begin at the pitch point, P and continue through a period depending upon the length of the teeth.

It is evident that a generator G' could be made to describe flanks for A and faces for B, as shown by the curves $a'g'$, and $b'g'$, respectively, which would satisfy the conditions of constant velocity ratio, and that the action of this pair of curves is entirely independent of the first pair; hence G and G' may be any two circles.

At the sides of the figure are shown complete teeth of A and B, the outlines of which correspond to the curves traced by the

describing circles G and G'. The faces of A and the flanks of B are the epicycloid and hypocycloid generated by G, and are identical in form with the curves ag and bg, respectively. The faces of B and flanks of A are of the forms generated by G', as shown by $b'g'$ and $a'g'$, respectively. The teeth are symmetrical; therefore either side may be the acting side, and either wheel may drive.

If the common pitch, p, is an exact divisor of both circumferences; if the lengths of the teeth are such that at least one pair shall always be in contact; and if the spaces are deep enough to allow the points to clear in passing the centre line, these wheels will meet all essential requirements.

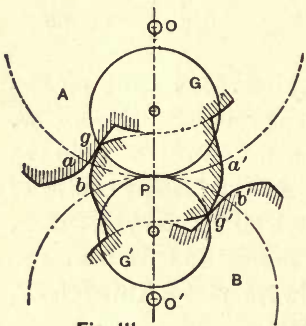

Fig. 111

63. Action of Epicycloidal Gear Teeth.—In Fig. 111 the tooth outlines at the left are just coming into contact, and those at the right are just quitting contact. The angle aOa' through which gear A turns while one of its teeth is in contact with a tooth of B, is called the *angle of action* of A. The angle aOP, passed through while the contact point is approaching the pitch point is the *angle of approach* of A, and angle POa' passed through while it is receding from the pitch point is the *angle of recess*. The angles of action, approach, and recess of B are $bO'b'$, $bO'P$, and $PO'b'$, respectively. The *path of the point of contact* during approach is along the arc gP of the describing circle G, and during recess it is along the arc Pg' of the describing circle G'.

It has been shown (Art. 62) that the arcs Pa, Pb, and Pg are of equal length. The arcs Pa', Pb', and Pg' are also of equal length. Therefore the *arcs of action*, aPa' and bPb', subtended by the respective angles of action are of equal length, and this length is equal to that of the path of the point of contact, gPg'.

When the point of contact is at P, the teeth have pure rolling action; at all other times the action is mixed sliding and

rolling (Art. 36). The rate of sliding is greatest when the point of contact is farthest from the pitch point, and it decreases to zero at the pitch point. Since this sliding causes friction it is desirable to reduce it to a minimum. Decreasing the length of teeth lessens the angle of action and the length of the path of contact, and therefore reduces the sliding. For continuous action, one pair of teeth must come into contact before the preceding pair quits contact; therefore, the angle of action cannot be less than the angle subtended by the pitch arc; or the arc of action aPa' (or bPb') must at least equal the distance between similar points (on the pitch line) of two adjacent teeth of either wheel. This condition fixes the minimum length of the teeth. If the given pitch of the two wheels (Fig. 111) is $aa' = bb'$, this determines the minimum arc of action. This arc may be distributed in any way between the approach and recess arcs, though these are commonly nearly equal. Lay off $Pa = Pb = Pg$, and $Pa' = Pb' = Pg'$ equal to the desired arcs of approach and recess, respectively; then g and g' are the extreme points in the faces of B and A, respectively; or circles drawn with the radii $O'g$ and Og' are the boundaries of the teeth of the two wheels. The strength of the teeth depends upon their thickness, and the pitch is ordinarily twice the thickness of the teeth at the pitch circle, or slightly greater to allow clearance at the sides, which is called "backlash;" thus the pitch is a function of the force to be transmitted. As has been shown, the arc of action must at least equal the pitch; it is often made great enough to insure that two teeth shall always be in contact; or that as one pair is in contact at the centre line, the preceding pair shall be quitting contact, and the succeeding pair shall be beginning contact. This requires an arc of action equal to twice the pitch arc, and correspondingly longer teeth, for a given pitch.

The force acting between the teeth is transmitted in the direction of the common normal (neglecting the effect of friction), or in a line through P and the contact point of the teeth. This contact point always lies in the describing circle G during approach, and

in G' during recess; hence it appears that the force transmitted is more oblique as the contact point is removed from P. The effect of this obliquity is to increase the pressure between the teeth and at the bearings, with a corresponding increase in the energy wasted through friction. The friction due to the sliding action of the teeth tends to increase the obliquity of the pressure between the teeth during approach and to decrease it during recess by the amount of the angle of friction. Consequently the action during recess is smoother than it is during approach. For this reason gears are sometimes made in which the action is confined to the angle of recess, in which case the driving gear has faces only, and the driven gear has flanks only. Wheels of smaller pitch have shorter teeth, other things being equal, and their action is smoother under the ordinary conditions because the contact point is always nearer the line of centres, where the rate of sliding of the teeth upon each other is less.

It will be seen that, for a given pitch, the length of teeth required for a given arc of action is less as the describing circles used are larger in diameter.

64. Determination of Describing Circles.—During contact the faces of the teeth of A act only upon the flanks of the teeth of B; similarly, the faces of B act only on the flanks of A; hence the form of the faces of one wheel does not affect that of its own flanks nor of the faces of the mating wheel. There is no necessary fixed relation between the two describing circles G and G'.

If the describing circle has a diameter equal to the radius ($\frac{1}{2}$ the diameter) of the pitch circle within which it rolls in tracing a hypocycloid, this special hypocycloid is a right line passing through the centre of the latter circle, or a diameter of it. Hence if the describing circles, G and G' (Fig. 110), have diameters equal to the radii of B and A, respectively, both wheels will have *radial flanks;* but these will operate properly in conjunction with the corresponding epicycloidal faces. The faces would not, in this case have the forms shown in Fig. 110, as the faces of one wheel and the flanks of the other one must be derived from equal describing circles. The

radial flank forms are simple in construction and describing circles are sometimes used for a pair of gears which will give such teeth. If the describing circle has a diameter less than the radius of the pitch circle within which it rolls in tracing the hypocycloid, the flanks lie outside of radii through the pitch point; while if the diameter of the describing circle is greater than the radius of this pitch circle, the hypocycloidal flanks lie inside of the radii to the pitch points. The first of the forms gives spreading flanks which are much stronger than the converging or undercut flanks of the latter form. The radial flank is intermediate between these forms in strength. Except in small gears (frequently called pinions) for light work, undercut flanks are seldom used; the radial flank usually being the weakest form allowed. While it is desirable for strength of the teeth to have spreading flanks, and therefore to use a small describing circle, large describing circles give teeth which act upon each other with less obliquity.

In exceptional cases, when a single pair of gears are to work together, it may be good practice to choose the largest pair of describing circles which will give the necessary strength of flanks, and flanks of a comparatively weak *form* may be used by giving a small excess to the pitch (thickness of teeth). In such cases of single pairs of gears, for reasons already given, radial flanks will sometimes be used for both wheels. In making a set of patterns (or cutters for cut gears), however, it is desirable on the score of economy to provide for the working of any wheel of the set with any other wheel of the same pitch. If this is possible the set is said to be *interchangeable*. Suppose that in the two gears, A and B (Fig. 110), the faces of the former and the flanks of the latter are generated by a describing circle G, and that the faces of B and the flanks of A are generated by another circle G'. It has been shown that these two wheels will work together. A third wheel, C, of the same pitch, can not work properly with *both* A and B; for if the faces of C are generated by G, and its flanks are generated by G', it may engage with B; but it can not act correctly with A, for the faces of A and the flanks of C are not generated by the same circle; neither

are the flanks of A and the faces of C, and the conditions of constant velocity ratio are not met by this construction.

If $G = G'$, C would work correctly with either A or B, or with any other wheel of the same pitch, the faces and flanks of which are epicycloids and hypocycloids generated on its pitch line by $G = G'$. We may then state that: *The conditions necessary in an Interchangeable Set of Gears are that all of the wheels of the set shall have the same pitch, and that the teeth of all of them shall have faces and flanks generated by equal describing circles.*

It is common to assume that the smallest wheel that will probably be required will be a pinion of either 12 or 15 teeth, and to take a describing circle which will give radial flanks to such a pinion; that is, a describing circle with a diameter half that of the pitch circle of this smallest pinion. If t is the number of teeth in the smallest pinion, its pitch-circle radius, or the diameter of the describing circle, $= \frac{tp}{2\pi}$.

65. Annular Wheels.—Fig. 65 shows two rolling circles, one of which is tangent to the concave side of the other. The corresponding rolling cylinders may be used as pitch surfaces of gears. The larger of these is called an *annular gear*.

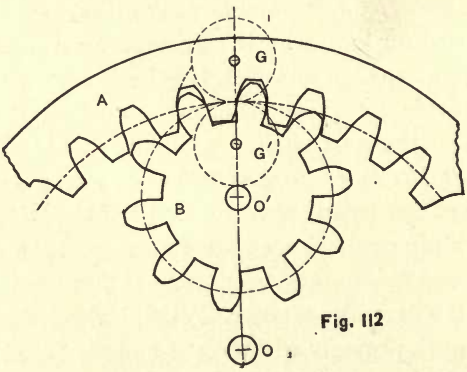

Fig. 112

The method of generating the teeth of such gears is indicated in Fig. 112, and it is similar to that explained for external gears,

except that the faces of B and flanks of A (generated by G) are *both epicycloids*, and the faces of A and the flanks of B (generated by G') are *both hypocycloids*.

66. Rack and Pinion.—If one pitch line is a right line (a circle of infinite radius), as shown in Fig. 113, teeth may be formed by a method similar to that given for the more general case of spur gearing. Such a gear is called a rack, and the wheel which meshes with it is usually called a pinion. The faces and flanks of the rack are *both cycloids ;* and they are alike in an interchangeable set of gears, where but one describing circle is used. In such a set, any wheel will engage properly with the rack. The construction of teeth for a rack and pinion is shown at the left of Fig. 113 ; and at the right, the complete teeth are shown in the acting positions. Of course the rack is necessarily of limited length,

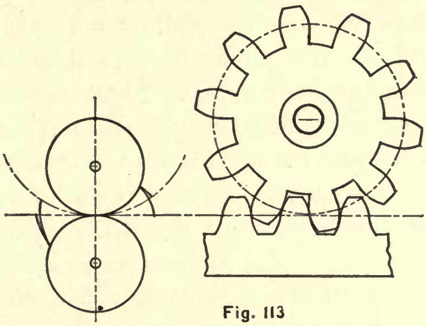

Fig. 113

and the motion transmitted between a rack and pinion must be reciprocating.

67. Pin Gearing.—If the describing circle equals one of the pitch circles, the hypocycloid in this pitch circle becomes a mere point; and this point acting on an epicycloid generated on the other wheel by this same describing circle will transmit a motion identical with the rolling of the two pitch circles. Fig. 114 shows such a point in B acting on the epicycloidal faces of A. In an actual gear a pin of sensible diameter must be used, and Fig. 114 shows such a pin, and dotted line curves parallel to the

original epicycloid of A and at a distance from this epicycloid equal to the radius of the pin. This pin and the dotted outline will transmit the same motion as that due to the point and epicycloid.

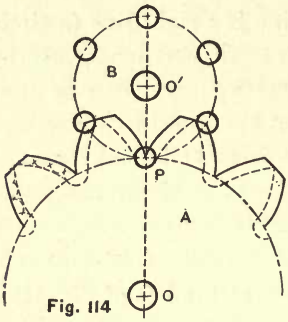

Fig. 114

With the point and the epicycloid the angle of action is entirely on one side of the line of centres, and the pin gear should always be the follower, in order that the action shall take place during recess rather than approach. With a pin of sensible diameter the action begins at a distance, practically equal to the radius of the pin, before the line of centres is reached, and there is consequently also an angle of approach. The derived curve of the driver gives shorter teeth than the full epicycloids, and the height of the driver's teeth, above the pitch line, is therefore diminished, thus decreasing the angle of recess. These gears were formerly much used, when teeth were commonly made of wood, as the pin form is easily constructed; but this class of gearing is now used but little, except for light gearing, such as clockwork, etc.

68. Involute Teeth.—When a right line rolls on the circumference of a circle any point of the line traces an involute of the circle. It is a property of this curve that the normal at any point is tangent to the base circle. In Fig. 115, if the right line EE', tangent to the base circles aa and bb, has rolling contact with these circles as they rotate about the fixed centres O and O', respectively, any point g of EE' traces an involute of each base circle.* In every phase of this operation these two involutes are tangent to each other at the position of g, and the line EE' is

* The right line EE may be considered as the tangent portion of a flexible band which wraps upon one base circle and unwraps from the other, as they rotate. The point g in this band generates the two involutes upon the rotating planes of the respective circles.

normal to both curves at this point. This common normal always cuts the line of centres, OO', in a fixed point, P. If these involute curves are used as the outlines of teeth for two gears A and B, turning about the fixed centres O and O' respectively, a constant angular velocity ratio, equivalent to pure rolling contact between two pitch circles tangent to each other at P, will be maintained as long as the involutes are in contact. When A turns in a clockwise direction, the first contact between the tooth curves occurs when

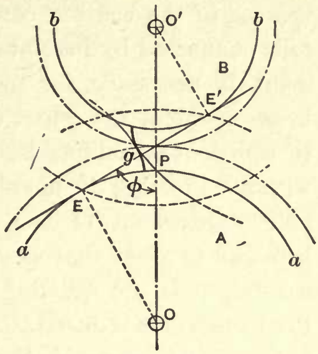

Fig. 115.

the tracing point is at E, and contact continues as g moves along the line EE' until E' is reached. For continuous action with teeth of involute outline the pitch angles must not exceed the angles through which the respective gears have turned during this time.

The line EE' is the *locus of the point of contact.* The angle between EE' and the common tangent to the pitch circle is the *angle of obliquity.* Standard gear cutters are so made that the sine of the angle of obliquity is 0.25, which corresponds to an angle of $14\frac{1}{2}°$.

It is an important property of involute tooth outlines that the distance between the centres of rotation may be changed without affecting the velocity ratio. Whatever the distance between the centres of the base circles of the two involutes, the common normal to both curves, in any position of tangency, is always tangent to both base circles, and divides the line of centres into segments which are proportional to the radii of the respective base circles. Since the angular velocity ratio is inversely proportional to the ratio of these segments, it is independent of the distance between centres.* This property is peculiar to

* It will be noted that the mathematical pitch circles vary in diameter with such adjustment of the centre distance, and the pitch circles of involute gears have not the same physical significance as in epicycloidal gears.

the involute system, and is exceedingly valuable, especially in in gears connecting roll-trains, in change gears, etc., where exact spacing of the centres can not be maintained. When a pair of rolls connected by involute gears becomes worn, or when adjustment is necessary for passing material of different thickness between them, the centre distance may be changed considerably (if sufficient initial backlash has been provided between the teeth) without affecting the angular velocity ratio. The limits of allowable adjustment of the centre distance are reached when it becomes so great that as one pair of teeth are engaging the preceeding pair are quitting contact, and when it is so small that the backlash is reduced to zero, on account of the greater thickness of the teeth inside the original pitch line.

The angles through which the gears may be turned while a a pair of involute tooth outlines are in contact depend on the angle of obliquity. When the angle of obliquity is zero, the base circles coincide with the pitch circles, and the involutes are both entirely outside the pitch circles, and can not come into contact except when passing the pitch point. The angle of action is zero. This represents a special (though impossible) case of epicycloidal gearing in which the diameter of the describing circles is increased to infinity. As the angle of obliquity is increased the angle of action also increases.

The length of teeth necessary for this action is indicated by the circles through E and E' with centres at O' and O respectively. Since the distance between either of these addendum circles and the corresponding pitch circle always exceeds that between the pitch circle and the base circle of the mating gear, it is necessary to extend the tooth spaces inside the base circles to accommodate the ends of the teeth. These extensions of the tooth outlines are usually radial lines tangent to the involute curves at the base line and to fillets at the roots of the teeth. Involute teeth are sometimes called teeth of single curvature, as there is not a reversal of curvature at the pitch line.

The tooth outlines of an involute rack are composed of straight lines perpendicular to the locus of the point of contact. Fig. 115*a* shows an involute rack and pinion in mesh.

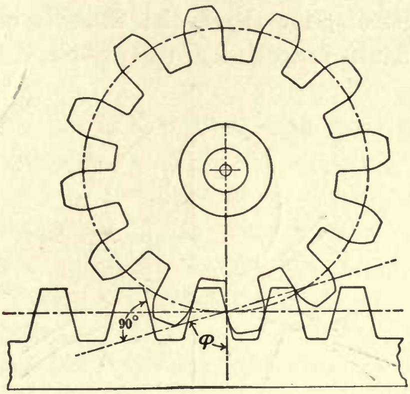

Fig. 115*a*

69. Interference in Involute Teeth.—The length of standard gear teeth is determined by their pitch, rather than by the considerations stated in the preceding article. When one gear has a small number of teeth of standard proportions the ends of the teeth of the mating gear extend beyond the point of tangency of the common normal and the base circle. This is shown in Fig. 116, which illustrates a pair of teeth in contact at the point of tangency between the base circle of the smaller gear *B* and the common normal. The part of the tooth outline of *B* inside the base circle is a radial line. It is evident that any further turning of *A* toward the right will result in contact with the radial part of the outline of *B*, and the angular velocity ratio will not be constant. This contact of the teeth inside the base circle is called *interference.* To avoid interference the flanks of the teeth of *B* may be hollowed out or the points of the teeth of *A* may be cut away. The latter is the usual remedy.

If the portion of the face which comes into contact with the radial flank of the mating tooth is given the form of an epicycloidal arc generated on the *pitch circle* by a describing circle of half the pitch diameter of the mating wheel, the action will be correct, for the radial flank is equivalent to a hypocycloidal flank formed by this same describing circle.

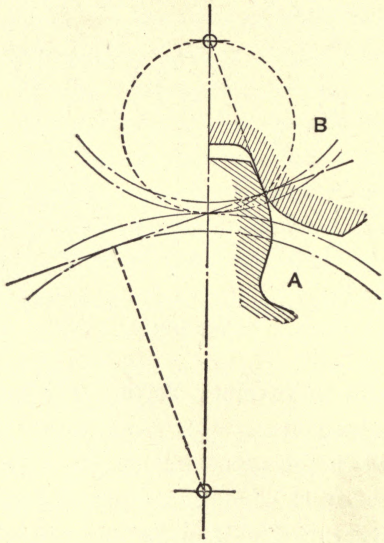

Fig. 116

This is strictly correct only if the given centre distance is maintained.

The least number of teeth that an involute gear of $14\frac{1}{2}°$ obliquity may have and mesh without interference with an equal gear is 23; the least number in a gear that will mesh with a rack without interference is 32.

70. Comparison of the Systems.—Since involute teeth transmit constant angular velocity ratio when the centre distance varies, exact setting is not so necessary, and wear of the bearings does not disturb the action as it does in the epicycloidal system.

The line of action is always in the same direction, and the force acting between the teeth is nearly constant in the involute system; while the acting force is variable both in direction and magnitude in the epicycloidal system. The former teeth wear more evenly as a consequence. The mean thrust on the bearings is slightly greater, but more uniform, with involute teeth.

All involute teeth have the same generator; hence the gears are interchangeable if of the same pitch. Epicycloidal teeth are better for low-numbered pinions, but otherwise have no great advantage and many disadvantages. They are, however, quite commonly used for gears having cast teeth; while the involute system has largely supplanted the epicycloidal system for cut gears.

71. Clearance and Backlash. The term backlash has already been explained as the clearance at the sides of the teeth; it is equal to the width of a space minus the thickness of a tooth, both measured on the pitch line.

The backlash provides for any irregularity in the form or spacing of the teeth. It may be very small in accurate cut gears; but must be larger in cast gears.

The spaces are always made deeper than is required to allow the points of the teeth to pass; this allowance is called bottom clearance, or simply clearance; it also provides a lodging-place for a moderate quantity of dirt or other foreign substance which may get between the teeth.

72. Pitch of Gear Teeth.—The action of the teeth is smoothest when the contact point is near the line of centres; hence a large number of small teeth gives more uniform action than fewer and larger teeth. The teeth must be thick enough to sustain the load, however, and the pitch is determined by this consideration. The formula $W = spfy$, known as the Lewis formula, is in general use for determining the load that may be carried by the teeth of gears. In this formula W = the total

load transmitted by the teeth, in pounds; s = the safe working stress allowed in the teeth, in pounds per square inch; p = the circular pitch, and f = the width of face, both in inches; y = a factor depending on the form of the teeth. For standard epicycloidal and $14\frac{1}{2}°$ involute teeth, $y = 0.124 - \frac{0.684}{n}$; where n = the number of teeth in the gear. The pitch of such teeth necessary to sustain a working load, W_1, per inch of width of face, for a gear of given diameter, D, as determined from the above formula is:

$$p = D\left(0.285 - \sqrt{0.081 - 4.60\frac{W_1}{sD}}\right). \quad . \quad . \quad . \quad (1)$$

The relation between the circular pitch, the number of teeth, and the pitch diameter of a gear is expressed by the equation, $pn = \pi D$, from which $p = \pi D \div n$, $D = pn \div \pi$, and $n = \pi D \div p$. When either the pitch or the diameter is taken of a convenient size, the other must be expressed as a decimal value. Thus the pitch diameter of a gear having 36 teeth of $1\frac{1}{2}''$ circular pitch is $\frac{36 \times 1.5}{\pi} = 17.19''$, while if the diameter is taken as $18''$ the pitch $= \frac{\pi \times 18}{36} = 1.571''$. It is much more convenient to express the tooth spacing in terms of the number of teeth per inch of diameter of the gear. This ratio is called the *diametral pitch*, for which the symbol p' is used. The relation between the diametral pitch, the number of teeth, and the pitch diameter of a gear is expressed by the equation $p' = n \div D$, from which $D = n \div p'$, and $n = Dp'$. For a gear $18''$ in diameter and having 36 teeth, $p' = 36 \div 18 = 2$.

The relation between diametral and circular pitch is found by combining the equations for the value of D. $D = pn \div \pi = n \div p'$. $\therefore p = \pi \div p'$, and $p' = \pi \div p$.

Expressed in terms of the diametral pitch equation (1) becomes

$$p' = \frac{s}{W_1}\left(0.194 + \sqrt{0.037 - 2.15\frac{W_1}{sD}}\right).$$

73. Proportions of Gear Teeth. — The dimensions of all other parts of gear teeth are usually expressed as functions of

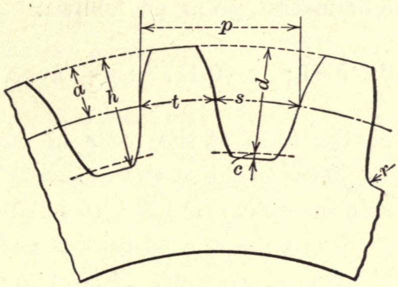

Fig. 117

the pitch. Fig. 117 will serve to explain the terms and symbols used for these parts.

p = circular pitch (measured on the pitch line).
t = thickness of tooth (measured on the pitch line).
s = width of space (measured on the pitch line).
a = addendum = length of tooth outside the pitch line.
d = working depth of tooth $= 2a$.
c = clearance.
h = whole depth of tooth $= 2a + c$.
b = backlash $= s - t$.
r = radius of fillet at root of tooth.

For gears having cast teeth the following are usual values: $t = 0.48p$; $s = 0.52p$; $b = 0.04p$; $c = 0.05p$; $a = 0.33p$; $d = 0.66p$; $h = 0.71\ p$.

The rims of such gears should be made about as thick as the base of the tooth not including the fillet. The hubs are usually about twice the diameter of the shaft. When the arms are of elliptical cross-section, the width of arm at the inside of the rim should be about $2\frac{1}{8}$ times the circular pitch. Such arms are tapered from $\frac{1}{4}''$ to $\frac{3}{8}''$ per foot on each side, and the thickness is made equal to $\frac{1}{2}$ the width.

The standard proportions of cut gear teeth are expressed in terms of the diametral pitch as follows:

$$a = 1'' \div p'; \quad b = 0; \quad c = 0.157'' \div p'; \quad h = 2.157'' \div p'.$$

The radius of the fillet at the roots of cut teeth is $\frac{1}{6}$ the width of the space between the teeth measured on the addendum circle. The outside diameter $= D + 2a = (n+2) \div p'$.

The above proportions are used for cut gears of the epicycloidal and $14\frac{1}{2}°$ involute systems. Where greater strength is desired without a corresponding increase in the pitch, involute teeth having an angle of obliquity of 20° are sometimes used. These teeth are called "*Stub Teeth*" because they are made shorter than teeth of standard proportions. In addition to being much stronger, these teeth have less sliding and no interference. On account of the greater obliquity of action the normal pressure between the teeth and the thrust on the bearings are greater for a given load than with standard teeth.

The pitch of stub teeth is usually expressed as a combination of two standard diametral pitches. Thus, a 4/5 pitch tooth has a thickness on the pitch line equal to that of a standard 4 pitch tooth, while the addendum is equal to that of a standard 5 pitch tooth. Other pitches are 5/7, 6/8, 7/9, 8/10, 9/11, 10/12, and 12/14. The depth of the space inside the pitch line is made $1\frac{1}{4}$ times the addendum.

74. Unsymmetrical Teeth.—If gears are always to turn in one direction, the opposite sides of the teeth may have different outlines. Fig. 118 shows such teeth. The working sides may belong to any system; the backs being so formed that they will not interfere, simply. Involutes, with an angle of the normal greater than would be practicable for *working* faces of teeth, are suitable for the backs. Stronger teeth, for any pitch, are obtained by this construction.

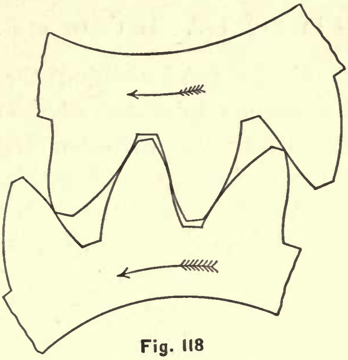

Fig. 118

Gear-teeth are seldom made unsymmetrical. Such forms will generally be more expensive, and the necessary strength may be obtained by increasing the pitch. If the force transmitted is excessive, and driving is always in one direction, teeth of this form may be used to avoid excessive pitch.

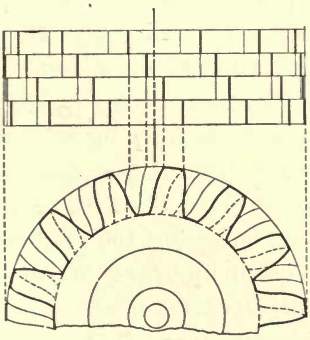

Fig. 119

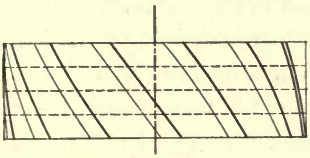

Fig. 120

75. Stepped and Twisted Gearing.—The action of gear-teeth is smoothest when the contact point is at the line of centres; for in this phase there is pure rolling between the teeth, and, in the epicycloid system, the obliquity is also zero at this instant. It is desirable to have the teeth as short as the required arc of action permits; but, as has been shown, this is governed by the pitch, which is a function of the force transmitted. It is therefore unsafe to reduce the pitch beyond a certain limit in a given case; but it is possible by "stepped" teeth to retain the required pitch, and still have a pair of teeth always in contact near the line of centres. Suppose a spur gear

to be cut by a series of equidistant planes, perpendicular to the axes; then let the slices into which the gear is divided be placed as in Fig. 119. If there are N of these slices, each may be $\frac{1}{N}$th the pitch ahead (or behind) the adjacent one. In this arrangement, the maximum distance of the nearest contact point from the line of centres is the corresponding distance with ordinary spur-gears of the same pitch and length of teeth, *divided by N.*

The thickness of the teeth has not been reduced by this modification, hence the strength has not been sacrificed. Large gears are sometimes constructed on this principle, with two sets of teeth, stepped one-half the pitch.

As the number of slices into which the gear of Fig. 119 is cut increases (their thickness decreasing correspondingly) the teeth approach those of Fig. 120, which represents the limiting form of the stepped wheels. That is, when the number of slices becomes infinite, the stepped elements become spirals. Gears with teeth of this kind are called *twisted gears.* It is to be noticed that the action of these twisted teeth is similar to that of the corresponding spur-gears, and they must not be confused with *screw-gears* which they resemble in form, but which are not constructed for parallel axes. The distinction will be considered more fully in a later article. With twisted gears there is a component of the pressure transmitted which tends to slide the wheel along the axis, or to crowd the shaft to which the wheel is attached against the bearing. This thrust against the bearing can be taken up by a collar, and axial motion thus prevented, but such an expedient results in an undesirable frictional loss, with risk of heating, etc. By twisting the teeth on the opposite sides of the central section in opposite directions, as shown in Fig. 120 (a), the axial efforts due to these two halves balance each other, and there is no such thrust imparted to the shaft. In actual gears of this form, the two halves may be cast in one piece, if the teeth are not to be

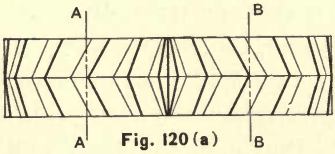

Fig. 120 (a)

machined; but if cut gears are used, the two halves are made as two separate gears of opposite inclinations (the elements of one half are right-handed helices, and of the other are left-handed helices), and these two gears may be attached firmly to the shaft, side by side, thus constituting practically one wheel.

It will be seen that these twisted gears always have one contact point on the line of centres, if the twist within the width (or the half width, with the double form of Fig. 120 (a)) of the gear is at least equal to the pitch, and the action at this one point is pure rolling. Now if the addenda of these teeth are relieved as indicated in Fig. 121 by the dotted lines, the faces will not act upon the flanks of the mating gear till the pitch point of any section comes into contact; that is, till the line of centres is reached, when the action is pure rolling. If the mating gear is similarly treated, the two gears only touch at a point, and this point is always in the line of centres. Thus contact for any tooth begins when the foremost section reaches the line of centres; it travels along the pitch element of the tooth ending as the last section passes the line of centres, the most advanced section of the next pair of teeth taking up the action in turn. This concentrates the force transmitted at a single point, theoretically, which may result in too intense a pressure in heavy work; but it has the effect of producing pure rolling between the teeth in contact at all times. *This is probably the only example of combined pure rolling, constant angular velocity ratio, and positive driving.*

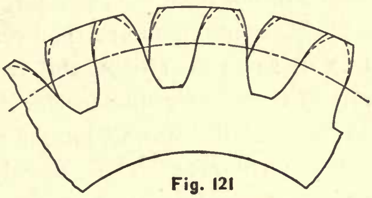
Fig. 121

76. Non-circular Gears.—In Arts. 46, 47, 48, and 49 the action of rolling ellipses, rolling logarithmic spirals, general rolling curves, and lobed wheels was briefly explained. It has been seen that cylinders (in the general sense) corresponding to these curves may roll together, and it has been stated that such surfaces may be used as pitch surfaces for non-circular gears.

The general method of describing gear-teeth, as given in Art

59, may be applied in designing teeth for such gears, but a convenient approximate method will be indicated, using the rolling ellipses for illustration. Circular arcs can be drawn which closely approximate the ellipse at any point, and the methods for circular pitch line gears can then be used for the teeth. Or accurate elliptical curves may be drawn ; then lay off the pitch upon them and apply the method given in Arts. 59 or 62, in generating the teeth on these arcs. Of course, with a given describing curve, the teeth on portions of the pitch line which have different curvature will not have the same form.

This approximate method can be applied to other rolling curves as well as to the ellipse, and it is thus possible to form teeth for any of the non-circular forms (including the lobed wheels of article 49) which will transmit motion similar to that due to the rolling of the pitch curves. The general method of Art. 60 may be used if preferred.

77. Approximate Methods of Constructing Profiles.—The exact construction of the tooth profiles is somewhat tedious, and in many practical applications simpler approximate outlines may be substituted. Gears with cast teeth, especially if the pitch is small, depart somewhat from the ideal form, however carefully the patterns may be made ; and, therefore, some one of the approximate methods is generally used for laying out the patterns. In making cutters for cut gears, the exact method is usually employed.*

The arcs of the curves (epicycloids and hypocycloids, or involutes) used in gear-teeth are so short that circular arcs can be found which very closely approximate these curves ; and most of the approximate constructions are circular-arc methods.

A method given in Unwin's Machine Design has the merit of requiring no tables or special instruments, and it will be described first. In Fig. 122, *A* and *B* are the two pitch circles, and *G* and

* A little book published by the Pratt & Whitney Co. gives a description of the machine used by this company for accurately making these cutters automatically. This treatise was written by Professor McCord, and is reproduced in his Kinematics.

G' are the describing circles. Ph is the height of the addendum of B, and Pe represents two thirds this height. If a circle is drawn through e with a centre at the centre of the pitch circle B, it cuts the describing circle G in g; and if the arc Pb, on the pitch circle B, is laid off equal to the arc Pg on the generating circle G,

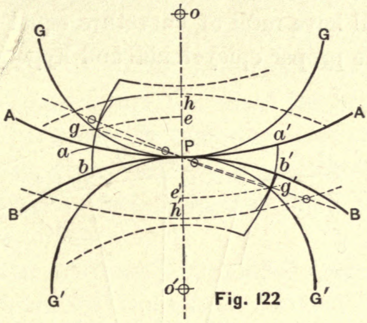

Fig. 122

b and g are two points in the tooth outline, as in the exact construction. Draw gP, the normal to the tooth profile at g. Now find by trial a circular arc with a centre on Pg, or its extension beyond P, which will pass through g and b. This arc passes through two points of the exact tooth outline, and its tangent at g also corresponds in direction with that of the true epicycloid, as the normal at this point is Pg for both the exact and the approximate curves. If the arc Pa is laid off on the pitch circle A, equal to the arc Pg, another circular arc, with a centre on the normal Pg, will pass through a and g, and it will approximate the flank of A. By a similar method, Pe' is laid off equal to two thirds the height of the addendum of A, and a circular arc with a centre on Pg', and passing through a' and g', is an approximation to the exact face profile of A. The approximate outline for the flank of B is an arc passing through b' and g', with a centre on $g'P$, or its extension. The point e need not necessarily be two thirds of Ph; but this fraction gives a good distribution of the error.

A method due to Professor Willis has been more widely used, perhaps, than any other. It may be briefly explained as follows:

Lay off from P (Fig. 123) the arc $Pa = Pa'$ = one-half the pitch, and draw radii Oa and Oa'; then draw the lines mm and $m'm'$ through a and a', making an angle ϕ with Oa and Oa' respectively. At a point c (on mm) take a centre, and draw a circular arc Pf through P; also with a centre at c' (on $m'm'$) draw the arc Pf' through P. The angle ϕ and the centres c and c' may be so chosen that these arcs will have radii of curvature equal to the mean radii of curvature of the proper epicycloidal and hypocycloidal faces and

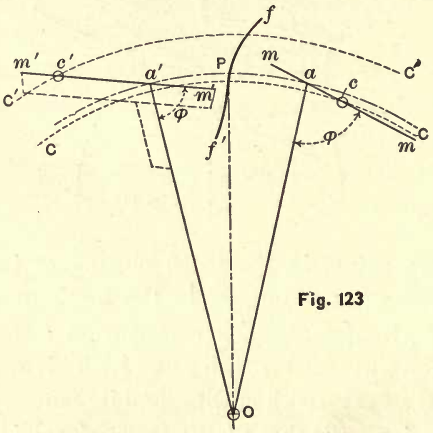

Fig. 123

flanks of the tooth. The angle ϕ was found by the originator of this method to give the best results when taken at 75°; $a \, . \, . \, c$ and $a' \, . \, . \, c'$ are given by the following formulas, in which p = pitch, and n = number of teeth in the wheel: $a \, . \, . \, c$ for faces $= \frac{p}{2}\left(\frac{n}{n+12}\right)$;

$a' \, . \, . \, c'$ for flanks $= \frac{p}{2}\left(\frac{n}{n-12}\right)$. An instrument known as Willis's Odontograph facilitates these operations. The form of this instrument is indicated by the dotted lines of Fig. 123. It is graduated along the edge $m'm'$ each way from the point a', and a table which accompanies the instrument gives the positions of the centres c and c' in terms of these graduations for wheels of given numbers of teeth and pitch.

Mr. George B. Grant has improved upon this odontograph by tabulating the radii for faces and flanks, and also tabulating the radial distances of the centres c and c' from the pitch circle. Mr. Grant has compiled a similar set of tables, called the Grant Odontograph, which are used in precisely the same way; but the values are different, giving somewhat different tooth profiles from those of the Willis system. The Willis method gives circular arc tooth outlines which are correct for one point, and which have radii equal to the *mean* radius of curvature of the exact curve. The approximate faces derived by this system lie entirely *within* the true epicycloids. Mr. Grant's system gives arcs which *pass through three points* of the exact profiles of the faces, and thus more closely approximate the correct curves.

The application of Grant's Cycloidal Odontograph, or, as its author calls it, from the method of deriving it, the Three-Point Odontograph, is shown in Fig. 124. The accompanying table is

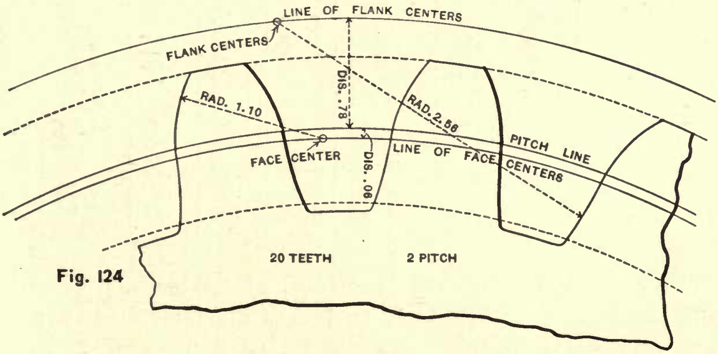

Fig. 124

given, together with his table for constructing approximate involute teeth. This matter is taken from Odontics, or the Teeth of Gears, by George B. Grant, and is reproduced by permission of the author.

To use the table in drawing approximate epicycloidal teeth proceed as follows: Draw the pitch line and set off the pitch, dividing the latter properly for thickness of tooth and space. The table gives values both in terms of One Diametral Pitch (equal 3.14″

circular pitch), and of One Inch Circular Pitch. Use the part of the table corresponding to the system of pitch employed.

THREE-POINT ODONTOGRAPH.

STANDARD CYCLOIDAL TEETH. INTERCHANGEABLE SERIES.

(From Geo. B. Grant's "Odontics.")

Number of Teeth.		For One Diametral Pitch. For any other pitch divide by that pitch.				For One Inch Circular Pitch. For any other pitch multiply by that pitch.			
		Faces.		Flanks.		Faces.		Flanks.	
Exact.	Intervals.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
10	10	1.99	.02	− 8.00	4.00	.62	.01	−2.55	1.27
11	11	2.00	.04	−11.05	6.50	.63	.01	−3.34	2.07
12	12	2.01	.06	∞	∞	.64	.02	∞	∞
13½	13–14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15½	15–16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17½	17–18	2.14	.11	6.13	2.20	.68	.04	1.95	0.70
20	19–21	2.20	.13	5.12	1.57	.70	.04	1.63	0.50
23	22–24	2.26	.15	4.50	1.13	.72	.05	1.43	0.36
27	25–29	2.33	.16	4.10	0.96	.74	.05	1.30	0.29
33	30–36	2.40	.19	3.80	0.72	.76	.06	1.20	0.23
42	37–48	2.48	.22	3.52	0.63	.79	.07	1.12	0.20
58	49–72	2.60	.25	3.33	0.54	.83	.08	1.06	0.17
97	73–144	2.83	.28	3.14	0.44	.90	.09	1.00	0.14
290	145–300	2.92	.31	3.00	0.38	.93	.10	0.95	0.12
∞	Rack	2.96	.34	2.96	0.34	.94	.11	0.94	0.11

The example of Fig. 124 is a wheel of 20 teeth, 2 diametral pitch, hence the pitch circle is 10 inches diameter (this figure is not reproduced full size). Opposite 20 teeth in the table we find .13 in the column of distances for faces ("dis."); divide this by the diametral pitch (2), giving .06″ as the distance of the circle of face centres from the pitch circle. Lay this distance off *inside* the pitch circle, and draw a circle through this point, concentric with the pitch circle. In a similar way the distance for flanks (1.57) is divided by 2, giving .78″, which is laid off *outside* the pitch circle, and a circle is drawn through this point. All tooth *faces* are to be

drawn with circular arcs having centres on the first of these lines of centres, and the flanks are drawn by arcs having centres on the last found line of centres. The tabular radius of faces for 20 teeth is given as 2.20, and dividing this by the diametral pitch, we get 1.10″ as the radius for the faces of the 2-pitch wheel. With this radius and centres on the face centre line, draw arcs through the proper points in the pitch circle, of course having the concave sides of the arcs toward the body of the teeth. In a similar way, the tabular radius for *flanks* (5.12) is divided by the diametral pitch, giving 2.56″ as the corrected radius. With centres on the flank centre line, draw arcs with this radius meeting the face arcs already drawn at the pitch point, and with the concave sides towards the spaces. Terminate the tooth profiles by the addendum and root circles, determined as in Art. 73, and put in fillets at the bottoms of the spaces.

If the circular pitch is used the construction is similar, using the appropriate portion of the table, but *multiplying* the tabular values by the circular pitch in inches instead of dividing.

As explained above, this table is calculated for arcs which pass through three points in the true curve. It is recommended that the student construct tooth profiles on a large scale by the exact method, and then draw the approximate profiles (superimposed), for comparison.

Grant's involute Odontograph given on page 142 is used as follows: Lay off the pitch circle, addendum, root and clearance lines, as in the preceding case. "Draw the base line one sixtieth of the pitch diameter inside the pitch line. Take the tabular face radius on the dividers, after multiplying or dividing it as required by the table, and draw in all the faces from the pitch line to the addendum line from centres on the base line. Set the dividers to the tabular flank radius (corrected), and draw in all the flanks from the pitch line to the base line. Draw straight radial flanks from the base line to the root line, and round them into the clearance line." [Grant's Teeth of Gears, p. 30.]

INVOLUTE ODONTOGRAPH.

STANDARD INTERCHANGEABLE TOOTH, CENTRES ON THE BASE LINE.

Teeth.	Divide by the Diametral Pitch.		Multiply by the Circular Pitch.		Teeth.	Divide by the Diametral Pitch.		Multiply by the Circular Pitch.	
	Face Radius.	Flank Radius.	Face Radius.	Flank Radius.		Face R dius.	Flank Radius.	Face Radius.	Flank Radius.
10	2.28	.69	.73	.22	28	3.92	2.59	1.25	0.82
11	2.40	.83	.76	.27	29	3.99	2.67	1.27	0 85
12	2.51	.96	.80	.31	30	4.06	2.76	1.29	0.88
13	2.62	1.09	.83	.34	31	4.13	2.85	1.31	0.91
14	2.72	1.22	.87	.39	32	4.20	2.93	1.34	0.93
15	2.82	1.34	.90	.43	33	4.27	3.01	1.36	0.96
16	2.92	1.46	.93	.47	34	4.33	3.09	1.38	0.99
17	3.02	1.58	.96	.50	35	4.39	3.16	1.39	1.01
18	3.12	1.69	.99	.54	36	4.45	3.23	1.41	1.03
19	3.22	1.79	1.03	.57	37–40	4.20		1.34	
20	3.32	1.89	1.06	.60	41–45	4.63		1.48	
21	3.41	1.98	1.09	.63	46–51	5.06		1.61	
22	3.49	2.06	1.11	.66	52–60	5.74		1.83	
23	3.57	2.15	1.13	.69	61–70	6.52		2.07	
24	3.64	2.24	1.16	.71	71–90	7.72		2.46	
25	3.71	2.33	1.18	.74	91–120	9.78		3.11	
26	3.78	2.42	1.20	.77	121–180	13.38		4.26	
27	3.85	2.50	1.23	.80	181–360	21.62		6.88	

Grant's special directions for drawing the teeth of the involute rack are substantially as follows: To draw the teeth for the involute rack, draw lines at 75° with the pitch line of the rack; the outer quarter of the tooth length (one half the addendum) is to be rounded off by an arc with a radius equal to 2.10″ divided by the diametral pitch, or .67″ multiplied by the circular pitch. This is to avoid interference.

78. Bevel-gears. - It was shown (see Art. 52) that a pair of cones can be placed on intersecting axes in such a manner that they will transmit motion with a given angular velocity ratio if they roll together without slipping. Such rolling cones may be used for pitch surfaces of bevel-gears just as rolling cylinders are used for the pitch surfaces of spur-gears, and teeth can be formed on these conical pitch surfaces which will transmit a positive motion equivalent to that of the rolling cones.

In treating of spur-gearing plane sections at right angles to the axes were used to represent the gear, and the tooth outlines were considered to be developed by the rolling of one plane curve (the describing line) upon another plane curve (the pitch line). The real teeth are, of course, solids bounded by ruled surfaces, all transverse sections of which are exact counterparts of the plane curves discussed. That is, the actual teeth are not lines generated by a point in the describing curve as it rolls upon the pitch line, but they are really surfaces generated by an element of the describing *cylinder* as it rolls upon the pitch *cylinder*.

Spur-gears coming under the head of plane motions permit representation by plane sections, as explained in Art. 10. This simple treatment cannot be applied to bevel-gears, for although each separate gear has a plane motion (rotation) about its axis, taking two cones rolling together, the relative motion is a spherical motion (see Art. 12). To construct bevel gear-teeth two projections are required.

Just as an element of one cylinder in rolling upon another cylin-

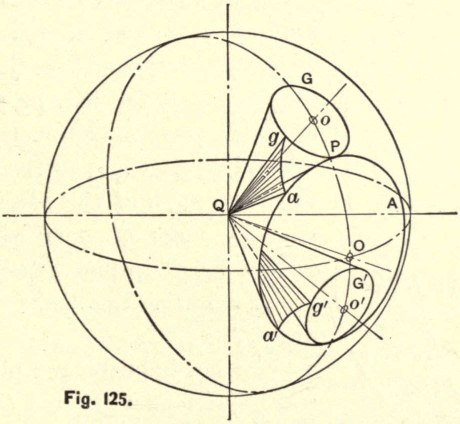

Fig. 125.

der generates a tooth surface, so an element of one cone in rolling upon another cone sweeps up a surface which can be used as the basis of a bevel-gear tooth. Fig. 125 shows a pitch cone A with the generating cone G (of equal slant height) in contact with it

along a common element PQ. All points in the bases of both cones are at the same distance from the common apex, hence these bases are small circles of a sphere which has a radius equal to the common slant height. If the generating cone be rolled upon the pitch cone, a point in the base of G, as g, will describe a curve on the surface of the sphere, relative to the base of the pitch cone. This curve is analogous to the epicycloid, the derivation of which was treated in Art. 62, and the curve now under consideration may be called a spherical epicycloid. In a similar manner another generating cone G' can roll inside the pitch cone, a point g' in its base tracing a spherical hypocycloid on the surface of the sphere. Points on other transverse sections of these generating cones would trace similar curves on spheres of different radii. A line passing through the centre of the sphere (the common apex of the cones) and moving along the spherical epicycloids and hypocycloids described as above, would give surfaces portions of which could be used as tooth boundaries. In other words, the elements of the describing cones which pass through g and g' would sweep up these surfaces. If two pitch cones of equal slant height have teeth generated in the manner just outlined, they will work together properly, transmitting a positive motion equivalent to the rolling of the pitch surfaces, provided the pitch of the teeth agree, and that the faces of each wheel are described by the same generating cone which describes the flanks of the other wheel.

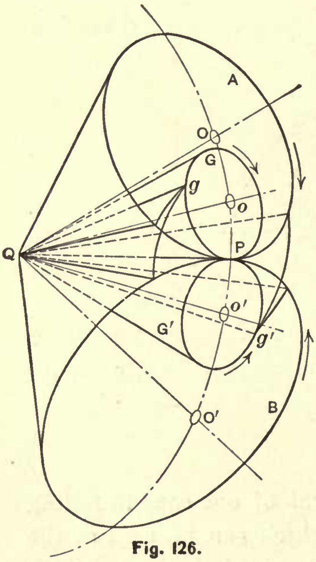

Fig. 126.

Fig. 126 indicates two pitch cones A and B, with axes QO and QO', respectively, and a common element of contact PQ. The describing cones are G and G', with axes Qo and Qo'; and if all four axes lie in one plane (a meridian plane of the sphere), they

can roll together about fixed axes, always having a common contact element in PQ. As the rolling proceeds, such an element of G as gQ sweeps up the faces for B and the flanks for A; and, at the same time, the element $g'Q$ of G' sweeps up the faces of A and the flanks of B.

The faces of B and the flanks of A always have gQ for a common element, and a plane through the three points PgQ, the common normal plane at the element of contact of these surfaces, always passes through the contact element of the pitch cones PQ. Likewise, the normal plane to the faces of A and the flanks of B, $Pg'Q$, always passes through PQ; hence teeth bounded by these swept-up surfaces will transmit motion with a constant angular velocity ratio. The analogy between this case and that of the spur-gear teeth, treated in Art. 62 (Fig. 110), is so close that further discussion is hardly necessary.

The describing surfaces are not necessarily right cones of circular cross-section, though these are the figures which correspond to the epicycloidal class of spur-gears, and are the only forms commonly employed. Any cone with an apex at the common apex of the pitch cones, and tangent to them along their common element might be used, as it would satisfy the kinematic requirements.

Involute teeth for bevel gears may be generated in a manner analogous to that used in Art. 68 for the generation of spur gear teeth.

It is difficult to construct spherical epicycloids and hypocycloids, and to represent the teeth of bevel gears on paper, and in practice a method know as Tredgold's approximation is always employed.

79. Tredgold's Approximate Method of Drawing Bevel-gear Teeth.—Fig. 127 shows the projection of two cones (with bases PM and PN) on a plane parallel to both axes QO and QO'. The line OO' is drawn perpendicular to the contact element PQ, then OM is drawn perpendicular to MQ and $O'N$ is drawn perpendicular to NQ. A cone can be constructed on the axis OQ with OP and OM as elements, and another on $O'Q$ with $O'P$ and $O'N$ as elements. These cones are called normal cones to A and B, respectively, as any element of one of these cones is perpendicular to an element of the pitch cone

having the same axis and the same base. The surfaces of these normal cones approximate the spherical surface for a short space either side of the pitch circles, and the conical surfaces have the practical advantage that they can be developed upon a plane for the construction of tooth profiles. Tredgold's approximate method consists in describing tooth outlines on these developed surfaces of the normal cones, and then wrapping these surfaces back to their original positions. The development of the normal cone surfaces is indicated in Fig. 127 by $PM'O$, and $PN'O'$. Upon the de-

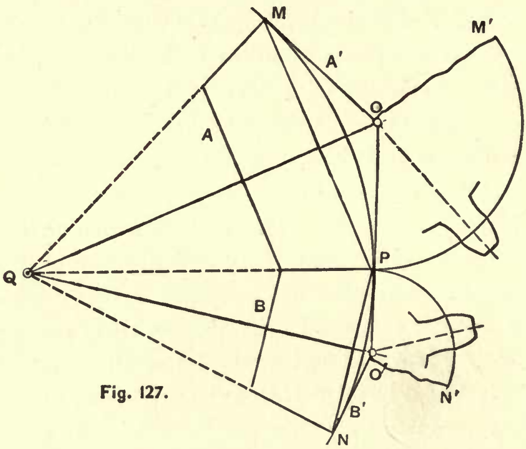

Fig. 127.

veloped bases of these cones (PM' and PN') as pitch lines, tooth outlines can be drawn by any of the methods used for spur-gears, just as if these were the pitch lines of such gears; and when these surfaces are rolled back into the normal cones the ends of the teeth are given by the profiles constructed in this way. A straight line passing through Q and following such a profile would sweep up tooth surfaces, all elements of which are right lines converging at Q. Within the limits of practice, such teeth, if properly constructed, agree quite closely with the exact forms.

The application of this method is shown in detail in Fig. 128 for the teeth of a single wheel. The pitch cone is shown in side elevation by MQM, and in plan by the circle M_1M_1. The side elevation of the normal cone is projected in MOM, and the develop-

ment of the pitch circle is show by MM'. The pitch, which must be an aliquot part of the pitch circle M_1M_1 ($= MM$ times π), is laid off on MM', and the addendum and root circles (AA' and RR', respectively) are drawn to give the proper length of teeth. The tooth outline is now constructed on MM' as for a spur-gear.

It is evident that all of the pitch points will fall upon the line MM, in the side elevation, when the normal cone surface is re-

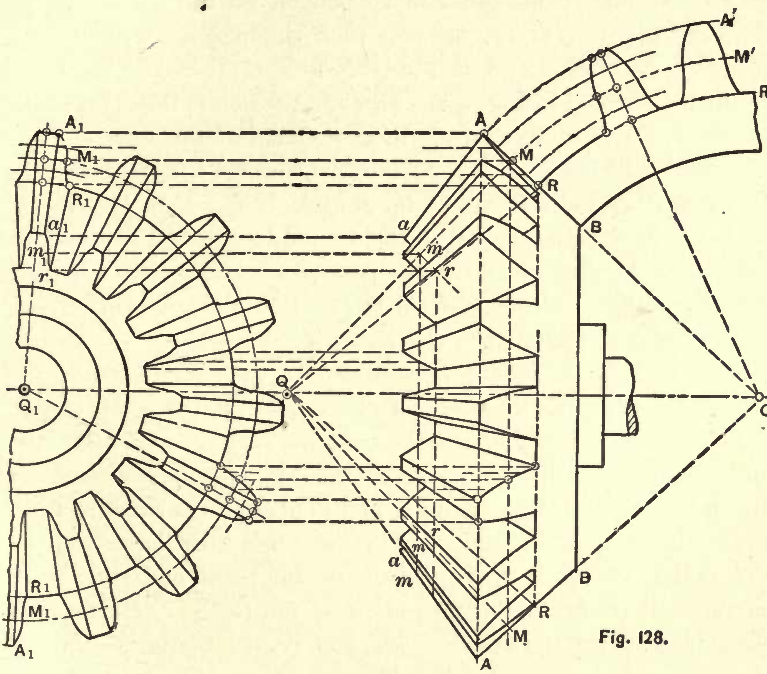

Fig. 128.

turned to its original position; that the outer ends of the teeth will fall upon AA; and that the bottoms will fall upon RR. In the other projection, the pitch points will all lie on the circle M_1M_1; the tops will fall on the circle through A_1A_1; and the bottoms will be in the circle R_1R_1. In this last projection (plan) the teeth will all appear the same, and they will have their true thickness at all parts; but the height (AR) will be shortened to A_1R_1. Divide up the circle M_1M_1 into the proper number of divisions for teeth and

spaces, and draw radii through the middle of the tooth divisions. Lay off half the thickness of the tooth at the pitch line (as obtained from the construction on developed pitch line MM') each side of these middle radii upon the circle M_1M_1; then lay off half the thickness of the top in a similar way on the circle A_1A_1; and half the thickness at the bottoms on the circle R_1R_1. The half thickness at positions intermediate between the pitch circle and the addendum or root circles can also be laid off on the corresponding circles in the plan, taking as many such thicknesses as the desired accuracy requires. The method of finding these intermediate points is indicated in Fig. 128. Through the points thus found, the curves $A_1M_1R_1$ can be drawn, giving the plan of the large ends of the teeth. To complete the plan of the wheel we may proceed as follows: Draw radii from A_1, M_1, R_1, etc., to Q_1; then lay off Aa, on the side elevation, equal to the desired length of the tooth, or the face of the gear, and draw amr parallel to AMR; from $a, m,$ and r, points lying in elements from Q to A, M, and R, respectively, carry lines across parallel to QQ_1; then circles with Q_1, as a centre and tangent to these several parallels are the projections in the plan of the addendum, pitch, and root circles for the small end of the teeth. As all elements of the teeth converge in plan at Q_1, the intersections of radii through A_1, M_1, and R_1 with these circles last drawn locate points a_1, m_1, and r_1 in the plan of the small ends of the teeth. Curves through these intersections, with the portions of the radial elements intercepted between them and the outer curves, will complete this projection of the teeth. Returning to the side elevation, the lines aa, mm, and rr, are the projections of the smaller addendum, pitch, and root circles. To complete the side elevation of the teeth project across (parallel to Q_1Q) from the various points on A_1A_1 to AA; from M_1M_1 to MM; from R_1R_1 to RR, etc., and draw curves through the intersections thus made. This will give the side elevation of the large ends of the teeth by passing curves through the corresponding intersections. In a similar way the side elevation of the smaller ends is obtained, and elements through Aa, Mm, Rr, etc., completes this view. It is

evident that each of the teeth appears in this projection with its own distinctive form.

The ring, or rim, which supports the teeth usually has a thickness equal to the roots of the teeth at the large ends, and this rim, with the hub, arms, etc., can now be drawn.

80. Peculiar Smoothness in Operation of Bevel-gearing.—Among spur-gears of an interchangeable system, those with the larger pitch circles will drive more smoothly, other conditions being the same. By referring to the bevel-gear of Fig. 128 it will be seen that there are 16 teeth on a pitch circle of radius Q_1M_1 (diameter $= MM$); but these teeth have profiles similar to those of a spur-gear with a pitch radius OM, equal to the slant height of the normal cone, and therefore the action of the bevel-gear would correspond to that of this larger spur-gear instead of to a spur-gear of diameter MM.

The actual pitch diameter of the bevel-gear is to that of the equivalent spur-gear (so far as smoothness of running is concerned) as $\sin AOQ : 1$; thus if the pair of bevel-gears on shafts at right angles are equal, angle $AOQ = 45°$, when

$$\sin AOQ : 1 :: \tfrac{1}{2}\sqrt{2} : 1 :: .707 : 1.$$

81. Non-interchangeability of Bevel-gears. — Bevel-gears are almost always made to work together in pairs, and it is not therefore of great importance to adopt a standard describing circle for all pairs of the same pitch. If two intersecting axes approach each other at a fixed angle, there is but one bevel-gear which will work properly with any other gear;* for a change in the angular velocity ratio involves a change in the direction of the contact element (Ac of Fig. 96), and hence a change in *both* pitch cones. A given bevel-gear could work with more than one other wheel if the inclination of the axes varied correspondingly, but this is a con-

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* Mr. Hugo Bilgram has produced sets of bevel-gears, by his gear-shaper, in which several different sizes of gears work correctly with a single gear, and all the axes make the same angle with the common driver. However, the pitch cones all of these gears do not have a common apex, although the teeth elements all converge to a common point. These gears are not, properly, of the common type of bevel-gears.

dition seldom met, and so these gears may generally be designed to work in pairs without regard to other gears.

The describing circles (if the epicycloidal system is used) may be so taken that both wheels will have radial flanks, which gives a simple form of teeth to construct, though this is not always desirable. For convenience of manufacture it is desirable to have a uniform system, usually; and when bevel-gears are cut with rotary milling cutters of the common type, standard cutters are used for various gears of the same pitch. This will be discussed in a later article.

In a great majority of cases requiring bevel-gears the axes are at right angles to each other, and "stock-gears" for such cases can frequently be obtained from gear-makers, if the proportions are not unusual. These stock-gears are generally much less expensive than gears made to order; but special gears are almost always required when the angle of the axes is other than 90°. When the two gears are equal (angular velocity ratio 1 : 1), the gears are called *Mitre Gears.*

82. Helical Gears.—If two cylinders are tangent to each other, they will have line contact when the axes are parallel, and point contact when the axes are not parallel. In the latter case the two elements (one of each cylinder) through the point of contact determine a plane tangent to both cylinders. The radii of both cylinders at the point of contact are perpendicular to this plane. When the cylinders rotate on their axes, each of the points in contact moves in the tangent plane in a direction at right angles to the axis of rotation. This is shown in Fig. 129, which represents a plan view of two cylinders, A and B, in contact at P; $a \dots a$ and $b \dots b$ being the axes of the respective cylinders. The angle between the axes of the cylinders is θ. The linear velocities of the points of A and B in contact at P are assumed to be v_1 and v_2 respectively. v_1 is represented by Pl, and v_2 by Pm. These velocities have a common component, v, represented by Pn, drawn through P perpendicular to lm. The angle between v and v_1 is α, that between v and v_2 is β. The components of v_1 and v_2, in

a direction perpendicular to v are Pt and Pt' respectively. The algebraic difference of these components represents sliding at the points of contact, while the common component represents rolling contact in a direction perpendicular to the sliding. It appears that the action of the tangent cylinders may be considered as a combination of sliding in a direction, tt', determined by the relative magnitudes of the linear velocities of the contact points, and rolling in a direction, Pn, perpendicular to the sliding.

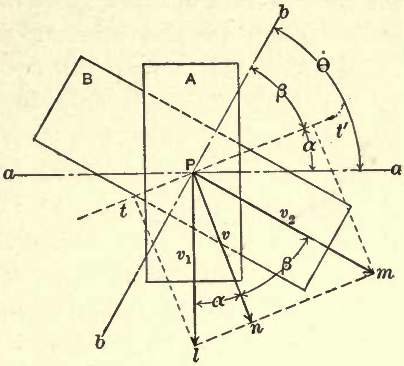

Fig. 129

Considering the rolling and the sliding separately, it is evident that, with a constant angular velocity ratio between the cylinders corresponding to the assumed linear velocities, the rolling action will result in rolling between two helices, each of which is called a *normal helix,* crossing the elements of the respective cylinders at angles of $90° - \alpha$ and $90° - \beta$, while the sliding is between helices crossing the elements at angles of α and β respectively.

If these cylinders are used as the pitch surfaces of gears to have an angular velocity ratio corresponding to the assumed values of v_1 and v_2 the teeth of these gears must be so formed as to permit the sliding action, and to transmit a velocity ratio corresponding to the rolling action. If the teeth are of uniform cross-section, and the pitch elements are helices making angles α and β with the elements of the respective pitch cylinders, the sliding may take place. The form of tooth outlines necessary to transmit constant velocity ratio equivalent to pure rolling of the normal helices is determined by the curvature of these helices, being the same as that for spur gears, the pitch lines of which have radii equal to the radii of curvature of the respective helices.

Such gears are called *helical gears.* They are included (together with worm gears, which are treated in the next chapter) under the general head of screw gears in the classification on page 110. Helical gears are also commonly known as *spiral gears.* They closely resemble *in form* the twisted gears described in Article 75, but their action is entirely different. It is to be noted that the axes of twisted spur gears are always parallel, while helical gears may be designed for any angle between the axes, although they are usually at right angles. The tooth action of helical gears is widely different from that of twisted gears. The former have point contact, and the latter line contact, while the screw-like action of helical gears results in a large amount of sliding in the direction of the common tangent to the tooth elements. This sliding action is entirely absent in the case of twisted gears. In twisted gears the angular velocity ratio is inversely proportional to the radii; while in helical gears it depends not only on the radii, but also on the relative values of the angles α and β.

83. The Pitch of Helical Gear Teeth.—Fig. 130 represents the pitch surface of a helical gear. The pitch elements of the teeth are shown crossing the cylinder elements at an angle ϕ, called the *angle of cut* of the teeth. The pitch p of these teeth is the distance between corresponding pitch elements of adjacent teeth measured along a normal helix. The distance between the same elements measured on a transverse section of the pitch cylinder is $p \div \cos\phi$. If there are n teeth, $\pi d = n \times p \div \cos\phi$

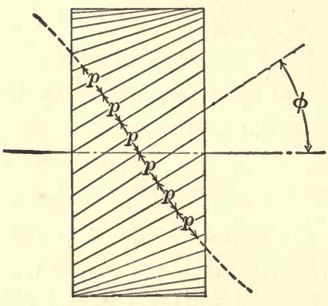

Fig. 130

or $d = n \div \frac{\pi}{p} \cos\phi$. Since $\frac{\pi}{p} = p' =$ the diametral pitch corresponding to p, the equation may be written $d = n \div p' \cos\phi$, or $n = d$

$\times p' \cos \phi$. The corresponding values for spur gears are $d = n \div p'$, and $n = d \times p'$.

84. The Velocity Ratio of Helical Gears.—If d_1 and d_2 are the respective diameters of A and B in Fig. 129 the corresponding angular velocities are $\omega_1 = v_1 \div \frac{d_1}{2}$ and $\omega_2 = v_2 \div \frac{d_2}{2}$. The angular velocity ratio is $\frac{\omega_1}{\omega_2} = \frac{v_1 d_2}{v_2 d_1}$. Since $v_1 \cos \alpha = v_2 \cos \beta$ this equation may be written $\frac{\omega_1}{\omega_2} = \frac{d_2 \cos \beta}{d_1 \cos \alpha}$. It is evident from the relation between the diametral pitch, the angle of cut and the number of teeth determined in the preceding article, that the number of teeth of A is $n_1 = d_1 p' \cos \alpha$ and the number of teeth of B is $n_2 = d_2 p' \cos \beta$. Substituting in the above equation, $\frac{\omega_1}{\omega_2} = \frac{n_2}{n_1}$, from which it appears that the angular velocity ratio of a pair of helical gears is equal to the inverse ratio of the number of teeth of the respective gears. This relation is the same as for spur and bevel gears, and all other classes of gears.

85. Outlines of Helical Gear Teeth.—It was stated in Art. 82 that, for constant angular velocity ratio, the tooth outlines of helical gears must have the same shape as those of spur gears having radii equal to the radii of curvature of the normal helices. The radius of curvature of a helix crossing the elements of a cylinder at any angle is equal to that at the end of the minor axis of the ellipse in which a plane making the same angle with the axis of the cylinder cuts the surface. In Fig. 131, d is the diameter of the pitch cylinder of a helical gear; *sos* is a tooth element; *non* is a normal helix; ϕ is the angle of cut. The major axis of the ellipse cut from the cylinder by a plane making the angle ϕ with a transverse section of the cylinder is $d \div \cos \phi$. The radius of curvature at the end of the minor axis is $\left(\frac{d}{2 \cos \phi}\right)^2 \div \frac{d}{2} = \frac{d}{2 \cos^2 \phi}$. Hence the tooth outlines of the helical gear should be the same

as those of a spur gear the diameter of which is $d \div \cos^2 \phi$. The number of teeth of a helical gear is $n = d\,p' \cos \phi$, that of a spur gear of diameter $d \div \cos^2 \phi$ is $n' = \frac{d}{\cos^2 \phi} p'$. $\therefore\ \frac{n'}{n} = \frac{1}{\cos^3 \phi}$, from which it appears that the teeth of a helical gear having n teeth

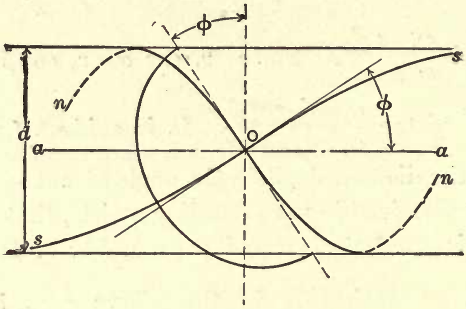

Fig. 131

should have the same outlines as those of a spur gear having $n \div \cos^3 \phi$ teeth.

86. Graphical Method for Helical Gears.—The relations between the dimensions of any pair of helical gears may be shown graphically as in Fig. 132. The angle between the axes $= xoy = \theta$. $xy = d_1 + d_2 = 2c =$ twice the distance between shaft centres; angle $oxy = 90° - \alpha$, and angle $oyx = 90° - \beta$; $gx = d_1$ and $gy = d_2$; gh and gk are perpendicular to ox and oy respectively; gb is parallel to ox, and ga is parallel to oy.

The following relations are evident:

$$(90° - \alpha) + (90° - \beta) + \theta = 180°; \qquad \therefore\ \alpha + \beta = \theta.$$

$$\text{Angle } hga = \text{angle } kgb; \qquad \therefore\ gh : gk :: ga : gb.$$

It has been shown (Art. 83) that $d_1 = n_1 \div p' \cos \alpha$, and $d_2 = n_2 \div p' \cos \beta$. In Fig. 132, $d_1 = gh \div \cos \alpha$, and $d_2 = gk \cos \beta$. Equating the respective values of d_1 and d_2,

$$gh = n_1 \div p', \quad \text{and} \quad gk = n_2 \div p'.$$

From these equations it appears that gh and gk are respectively equal to the pitch diameters of a pair of spur gears, each of which

has the same pitch and number of teeth as the corresponding helical gear. Combining these equations,

$$\frac{gh}{gk}=\frac{n_1}{n_2}=\frac{\omega_2}{\omega_1}; \text{ also } \frac{gh}{gk}=\frac{ga}{gb}=\frac{ob}{oa}; \quad \therefore \frac{\omega_1}{\omega_2}=\frac{oa}{ob}.$$

Also $xy=2c=d_1+d_2=n_1\div p' \cos\alpha+n_2\div p' \cos\beta$. Since $\beta=\theta-\alpha$, this equation may be written $n_1\div\cos\alpha+n_2\div(\cos\theta-\alpha)=2p'c$.

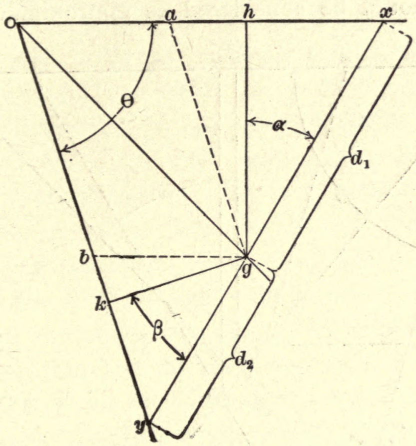

Fig. 132

A similar diagram (Fig. 133) may be constructed when only the centre distance, c; the shaft angle, θ; the velocity ratio, $\omega_1\div\omega_2$; and the diametral pitch, p', are given. From this diagram the values of n_1 and n_2, d_1 and d_2, and α and β may be determined.

Draw ox and oy making the angle θ at o. Lay off oa and ob on ox and oy, respectively, so that $oa : ob :: \omega_1 : \omega_2$.

Draw ae and be parallel to oy and ox respectively, intersecting at e. Draw oe. Draw the line xy, equal in length to $2c$, intersecting oe at g, so that the distance og is a maximum. Draw gh and gk perpendicular to ox and oy respectively. Multiply the lengths of gh and gk by the diametral pitch, p', to obtain approximate values of n_1 and n_2. For the actual values of n_1 and n_2 take the largest whole numbers, not greater than the approximate

values, which will satisfy the equation $n_2 \div n_1 = \omega_1 \div \omega_2$. Compute the corresponding values of $n_1 \div p'$ and $n_2 \div p'$, and take $h'g'$ and $k'g'$ equal to these values and parallel to hg and kg respectively, locating g' on oe. Through g' draw $x'y'$ equal in length to xy, terminating in ox and oy at x' and y' respectively. Then angle $x'g'h' = \alpha$, and angle $y'g'k' = \beta$; distance $g'x' = d_1$, and $g'y' = d_2$. Since it is not possible to obtain exact values by construction, these values should be considered as approximate. The exact

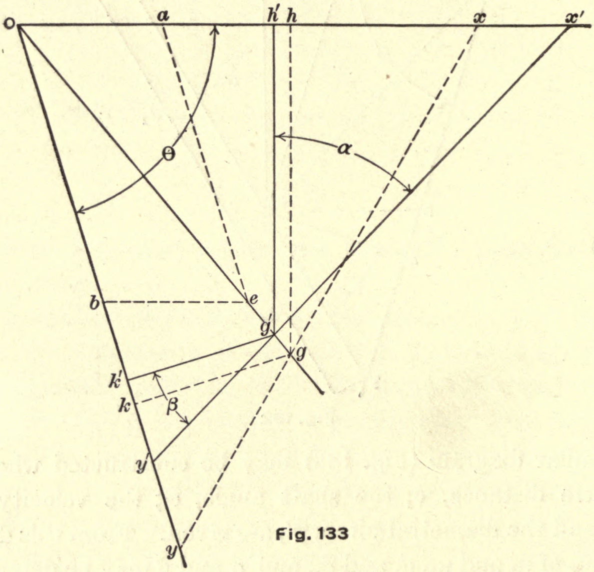

Fig. 133

value of α is obtained by trial from the equation $n_1 \div \cos\alpha + n_2 \div \cos(\theta - \alpha) = 2p'c$, using the approximate value first and changing it slightly until an exact equality results. The corresponding exact value of $\beta = \theta - \alpha$; those of d_1 and d_2 are obtained from the equations $d_1 = n_1 \div p' \cos\alpha$, and $d_2 = n_2 \div p' \cos\beta$.

When the shafts are at right angles, as is usually the case, $\theta = 90°$, and $\cos(\theta - \alpha) = \sin\alpha$. Substituting this value, the above equation becomes $n_1 \div \cos\alpha + n_2 \div \sin\alpha = 2p'c$. This may be written $n_1 \tan\alpha + n_2 = 2p'c \sin\alpha$.

It will frequently be found that some of the values determined by this construction are not suitable for actual gears. The next less number of teeth in each gear that will give the required velocity ratio may then be tried. If this does not give satisfactory values, either the pitch or the distance between centres must be changed.

87. Cast Gears.—Gear-teeth are either cut in a machine or are cast. For the rougher classes of work, it is common practice to use gears with cast teeth; but cut gears are now used almost exclusively for the better grades of work. When gears are cast, it is important to form the patterns very carefully, and especially to space the teeth accurately. With the utmost care, however, it is impossible to get very smooth and accurately spaced teeth, so clearance between the sides of the teeth, or backlash, must be provided. With small gears the enlargement of the mould, due to "rapping" the pattern, more than compensates for the shrinkage, and unless this is looked after in the pattern shop and foundry, the teeth may be too thick when cast.

A convenient device for forming the teeth of the pattern is shown in Fig. 134. A block of hardwood (preferably of a color quite distinct from the wood of the pattern) is shaped as shown, so that the sections at *amr* and *AMR* correspond to the two ends of the teeth (for a spur-gear these sections are, of course, alike). The middle portion is cut out, so that the distance L equals the length of a tooth; that is, the part removed corresponds to the form of a tooth. The stock for the teeth of the pattern is gotten out in lengths equal to L, and large enough in cross-sections to make a tooth. The block of Fig. 134 is screwed in the vise, and it may have two pointed brads projecting upward through the bottom. Then a piece of the prepared stock is forced down into the space

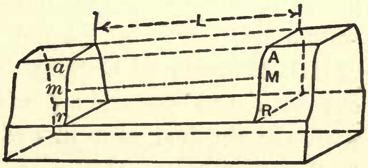

Fig. 134.

in this "form," and is then planed up with "hollow and round" planes. By working down to the form it is quite easy to produce a large number of teeth very uniform in shape.

Where many large cast gears are made, a gear moulding machine is sometimes used, as it produces accurate work and reduces the cost of patterns. A stake, or arbor, is set upright at the centre of the mould, which may be swept up in loam to approximately the outside form of the wheel. The pattern for the teeth, simply a block with a few teeth attached which corresponds to a segment of the entire rim, is fastened to the stake by an arm. This arm holds the segmental pattern at the proper distance from the axis of the wheel, and the arm and pattern can be turned about this axis. The pattern can also be withdrawn towards the centre or upward. In moulding the gear, this segment is placed in position and a few teeth are moulded by filling in about the pattern with the sand. The pattern is then drawn, rotated about the axis through a small angle (one or two teeth less than the number in the pattern), and a few more teeth are moulded. In this way the entire rim is moulded by sections. An index plate, or ring, is used to insure accurate spacing. After completion of the rim the pattern (or the cores) for the arms, hub, etc., is used to complete the mould.

88. Methods of Cutting Gear-teeth.—When a gear having cut teeth is to be made a *gear-blank* is prepared which is identical with the finished gear in every respect, except that it has no teeth. The teeth are then formed in this blank by cutting out the spaces between them. This may be done either by planing or milling. In planing machines the tool has a reciprocating motion, and cuts during the stroke in one direction only; in milling machines rotary cutters are used, and the cutting is continuous during the process of forming a space. In Fig. 135, *a* illustrates a tool used for planing gear-teeth, and *b* shows a standard milling-cutter. The cutting edges of both of these tools are formed to the exact shape of the space between two

teeth of the gear to be cut. Another class of tools have cutting edges formed to the shape of the tooth outline of another gear of the same pitch as the one to be cut. An important difference between these two classes of tools is that while the range of the number of teeth in a gear that may be cut with a single cutter of the first class is very narrow, any gear from the smallest pinion to a rack may be cut with a single cutter of the second type. On the other hand, it should be noted that the former class of tools may be used in the ordinary milling and shaping machines, without any special attachments other than an index-head for rotating the blank into the correct positions for cutting the teeth, while the latter class is used, in general, only in machines designed especially for cutting gears.

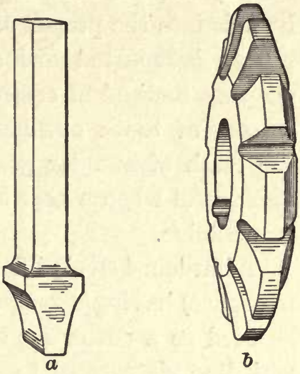

Fig. 135.

To reduce the wear of the finishing tool, it is customary (except in fine pitch gears) to roughly form the teeth by "gashing" the blank before the final cutting operation. When the teeth of a gear have been thus roughed out, they may be finished by planing with a tool ground to a sharp point which cuts the tooth surfaces element by element.

In the following articles the use of these methods in cutting spur, bevel, twisted and helical gears will be briefly outlined.

A detailed description of these operations and of the machines which perform them may be found in a book entitled "Gear Cutting Machinery" by Mr. R. E. Flanders.

89. Planing Spur-gears.—When the teeth of a spur-gear are to be cut with a tool such as that shown in Fig. 135*a*, the gear-blank is mounted between the centres of an indexing mechanism, placed on the table of a planing or shaping machine. The stroke of the tool is parallel to the axis of the gear. Between

strokes the tool is fed radially toward the axis of the blank until the required depth of space is obtained. The tool is then withdrawn, and the indexing mechanism used to turn the blank on its axis into the proper position for cutting the next space. This process is repeated until all the teeth are completed. This is the simplest method of cutting gear teeth. It may be used for any system of tooth outlines. It is especially adapted to cutting the teeth of annular gears, and gears of large diameter. It is also useful when a gear is needed and no standard milling-cutter is available.

A hardened steel pinion (properly "backed off" to give cutting clearance) having a reciprocating motion parallel to its axis may be used as a cutter for spur-gears. The gear-blank is mounted with its axis parallel to that of the cutter. The cutter and the blank are connected by a train of mechanism so that as the cutter is rotated very slightly on its axis between strokes, the blank also turns on its axis with a velocity ratio corresponding to pure rolling contact between the pitch-cylinders. The cutter thus meshes with the gear it is cutting as shown in Fig. 136. At the beginning of the operation of cutting a gear by this method the cutter is fed toward the blank until the distance between the axes corresponds to tangency of the pitch-cylinders of the cutter and the blank. The rotation on the axes is then begun. All the teeth are completed when one rotation of the blank on its axis has been made. A single cutter serves to cut all gears of a given pitch.

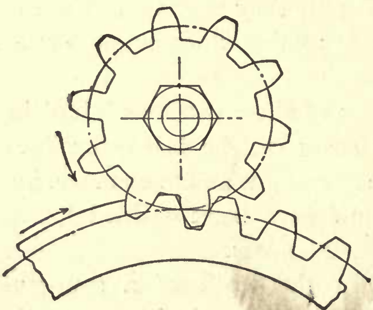

Fig. 136.

A tool having a cutting edge shaped like a single tooth of a rack of the same pitch as the gear to be cut may also be used in a somewhat similar manner. In this case the slight rotations of the

blank on its axis are accompanied by a motion of the tool in a direction at right angles to the stroke, the relative motion corresponding to pure rolling between the pitch surface of the gear being cut and that of the imaginary rack of which the cutter is a tooth. Fig. 137 shows the position of the tool relative to the space being cut at several stages during the operation. By indexing the blank after every stroke of the cutter, equal cuts are taken on all of the teeth. The rolling action between the pitch surface of the gear and that of the imaginary rack is independent

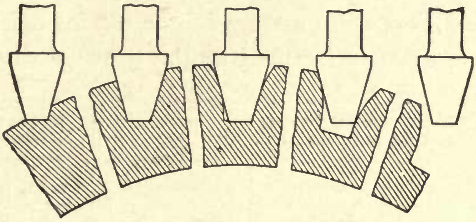

Fig. 137.

of the indexing, and occurs each time the blank has been indexed through a complete revolution.

Spur-gear teeth which are too large to be cut with formed tools may be finished with a planing tool having a sharp point, after the spaces have been roughly cut out by other means. In this operation the feeding of the tool toward the axis of the blank is accompanied by a lateral movement at right angles to the feed, causing the cutting point to trace the outline of the tooth. In this way the teeth are formed element by element. The lateral movement is usually produced by a templet formed to the exact shape of the tooth outline. After one side of one tooth has been completed the blank is indexed and the operation repeated on the next tooth. By providing a second tool to work simultaneously on the other side of the teeth, the necessity for turning the blank over after all the teeth have been finished on one side is avoided.

90. Milling Spur-gears.—When a standard milling-cutter, such as is shown in Fig. 135*b*, is to be used to cut spur-gear teeth, the blank is mounted with its axis at right angles to the axis of rotation of the cutter. The blank is fed toward the cutter and the whole space between two teeth is cut out by one passage of the cutter across the face of the blank. The blank is then turned into position for cutting the next space, and the operation repeated until all the teeth are completed.

Standard cutters are made in sets suitable for cutting all gears of a given pitch from a 12-tooth pinion to a rack. The following table shows the cutters in one set for epicycloidal and involute teeth, as manufactured by the Brown & Sharpe Mfg. Co.

Epicycloidal System 24 Cutters in each Set.				Involute System 12 Cutters in each Set.	
Cutter.	Teeth	Cutter.	Teeth.	Cutter.	Teeth.
A cuts	12	M cuts	27 to 29	1 cuts	135 to rack
B "	13	N "	30 to 33	2 "	55 to 134
C "	14	O "	34 to 37	3 "	35 to 54
D "	15	P "	38 to 42	4 "	26 to 34
E "	16	Q "	43 to 49	5 "	21 to 25
F "	17	R "	50 to 59	6 "	17 to 20
G "	18	S "	60 to 74	7 "	14 to 16
H "	19	T "	75 to 99	8 "	12 to 13
I "	20	U "	100 to 149		
J "	21 to 22	V "	150 to 249		
K "	23 to 24	W "	250 or more		
L "	25 to 26	X "	rack		

For absolute accuracy a different cutter would be required for each number of teeth of each pitch. Practically this is not necessary, as the change in the form of the teeth for a small change in number is slight, except in case of gears having few teeth. The form of teeth changes more rapidly for epicycloidal than for involute teeth. For this reason a larger number of cutters is required in an epicycloidal set.

When a large number of duplicate spur-gears are to be produced, the teeth are usually milled with a hob similar to that

shown in Fig. 154*a*. When the helix angle of the thread of the hob is ϕ, the hob is mounted with its axis of rotation making the angle $90°-\phi$ with the axis of the gear-blank. The blank and the hob are connected by a train of gears so that as the hob rotates the blank turns on its axis. The velocity ratio of the hob and blank is equal to the number of teeth of the gear to be cut, divided by the number of threads of the hob. The hob is fed slowly in a direction parallel to the axis of the blank, and

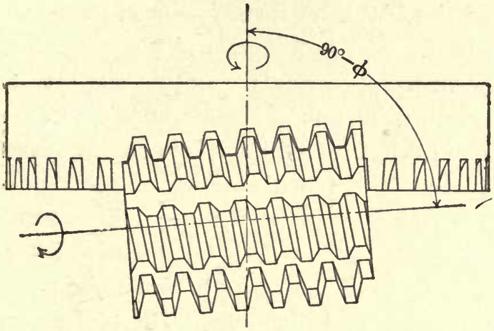

Fig. 138.

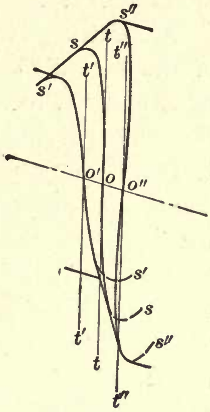

Fig. 138*a*.

as it cuts across the face of the blank the spaces between the teeth are cut to their full depth, all the teeth being completed by a single passage of the hob across the face of the blank. Fig. 138 shows the relative positions of the hob and blank in an early stage of the cutting. Teeth cut by this method are not theoretically accurate on account of the interference between the threads of the hob and the teeth of the gear. This interference is due to the fact that the helix angle of the thread elements of the hob is less than ϕ when these elements are outside the pitch-surface, and greater than ϕ when they are inside the pitch-surface. In Fig. 138*a*, the pitch element of a hob-thread is represented by *ss*, while $s's'$ and $s''s''$ are thread elements respect-

ively inside and outside of the pitch-surface. The corresponding tooth-elements of a spur-gear being cut by the hob are tt, $t't'$ and $t''t''$, all of which are parallel. While the pitch elements ss and tt are tangent at o, $s's'$ and $t't'$ intersect at o' and $s''s''$ and $t''t''$ intersect at o''. This interference, which is greatly exaggerated in the figure, tends to undercut the flanks and to cut away the outer ends of the teeth. The use of a single-thread hob of comparatively large diameter reduces the interference to a minimum.

91. Cutting Twisted, Helical, and Bevel-gears.—If the motion of the tool across the face of the gear-blank is accompanied by a uniform rotation of the blank on its axis the elements of the teeth cut will be helical. The cutter axis must, of course, be set at the proper angle with the gear axis. In this way helical and twisted gears may be cut, using any of the methods described for spur-gears, the turning of the blank on its axis being produced by a suitable train of gears. The tooth outlines of gears produced in this way will, in general, not be theoretically correct. Twisted gears of correct tooth outline may be planed with a cutting edge formed to the shape of the space between the teeth of a spur-gear of the same pitch and diameter. The tool must, however, be formed to work in a helical groove. Accurate helical gears may be cut by planing with a tool having a cutting edge formed to the shape of the tooth of a rack of the same pitch. The method usually employed for helical gears is either milling with a standard cutter or hobbing. In both operations there is interference between the cutter and teeth of correct outline, so that the resulting teeth are not theoretically correct. The error is so small, however, that it can be neglected in most cases.

It has been shown (Art. 78) that the tooth elements of a perfect bevel-gear all converge to the apex of the pitch cone. It is therefore impossible to cut a bevel-gear having theoretically correct tooth outlines by any method using a tool formed to the

shape of the space between two teeth, as this space not only changes in width in passing across the face of the gear, but the tooth outlines, though similar, grow smaller as they approach the apex of the pitch cone. Milling-cutters are, however, much used for bevel-gears. To make the space between the teeth narrower at one end than at the other it is necessary to take at least two cuts for each tooth space. By shifting the axis of rotation between these cuts it is possible to make the pitch elements of the teeth pass through the apex of the pitch cone. The thickness of the cutter used should not exceed the width of the space at the small end; the outline of the edge usually corresponds to that of the large end of the teeth. The resulting teeth are too thick above the pitch line at the small end. This extra thickness may be removed by filing. Similar results are obtained when bevel-gear teeth are planed with a tool formed to the shape of the tooth space. This method should be used only for bevel-gears of rather narrow face.

Accurate bevel-gear teeth may be cut by means of a tool having a sharp point, the strokes of which are directed toward the apex of the pitch cone, while the tooth outlines are determined by templet guides, or some equivalent arrangement. This method can be used only when the teeth have been previously roughed out in the blank. By using two templets and two cutting points both sides of the teeth may be formed simultaneously.

Another method that produces accurate involute bevel-gear teeth is illustrated in principle in Fig. 139. This is a modification of the process of planing spur-gears with a tool having an edge formed to the shape of the tooth of an involute rack, and it depends on the fact that the tooth outlines of an involute crown-gear (a bevel-gear the pitch elements of which are perpendicular to the axis) are identical with those of an involute rack of the same pitch. The crown-gear *A* meshes with a master-gear *B* having the same pitch angle as the gear to be cut. The gear-blank *C*

having teeth roughly cut is carried on the same shaft as the master-gear, and is rigidly connected to the master-gear, except when it is being indexed. The pitch-cone of the blank is tangent to the pitch-plane of the crown-gear, and rolls on this plane when the master-gear and crown-gear turn on their axes. The tool *D* has a straight cutting edge, corresponding to one side of an involute rack tooth. The tool-slide (not shown in the figure) moves on a guide attached to the crown-gear so that

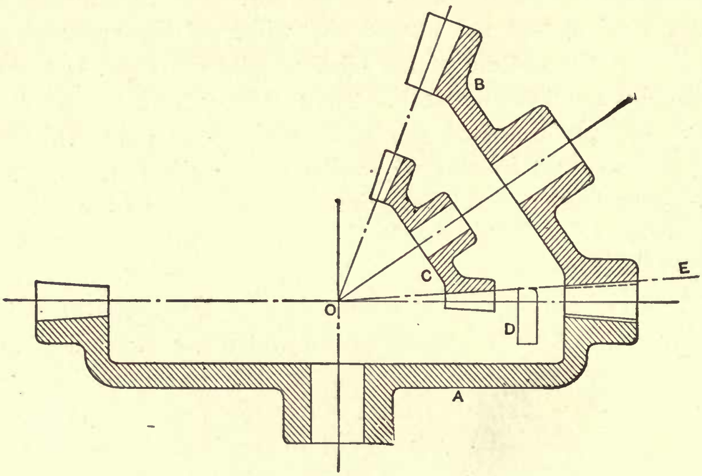

Fig. 139.

the edge of the tool describes one side of the tooth of an imaginary crown-gear meshing with the gear being cut. The motion of the point of the tool is toward the apex of the pitch cone, along the line *OE*. Between strokes of the tool the master-gear is turned through a slight angle, causing the crown-gear and the blank to turn slightly on their axes. By this means a pure rolling action is obtained between the pitch surfaces of the gear being cut and the imaginary crown-gear whose tooth is described by the edge of the tool. When one side of one tooth has been completed the blank is indexed on its axis into position

for forming the next tooth. Other mechanisms, giving identical motion, may be substituted for the crown-gear and master-gear shown in Fig. 139.

By the use of two tools both sides of the teeth may be formed simultaneously. The blank may be indexed after each stroke of the tool, instead of upon completion of a tooth. In this case the rolling action occurs on the completion of each rotation of the blank by the indexing mechanism.

Bevel gears may also be cut with a special hob, used in a special machine, but the operation is too complex to be described here.

92. Other Classes of Gearing.—In the table at the beginning of this chapter six classes of gearing are mentioned. Examples of all these classes except Skew and Face-gears have been discussed in this chapter. Skew-gears and Skew Bevel-gears are those based on rolling hyperboloids as pitch surfaces. These rolling hyperboloids were very briefly treated in Art. 53; but the theory of the teeth of these wheels is so complex, and their application is so very rare, that a discussion of them is hardly warranted in a short general treatise.

Face-gearing is an almost obsolete class, formerly used when wooden gears prevailed, because the teeth (mere pegs or pins) were easily made. The consideration of this class will also be omitted.

Twisted Bevel and Skew Gears are derived from the corresponding bevel and skew-gears in the same way that twisted spur-gears are derived from the ordinary spur-gears.

Worm-gears, which are a form of screw-gearing, will be treated in the next chapter.

The Practical Treatise on Gearing, published by the Brown & Sharpe Mfg. Co., gives a great many valuable points on the actual construction of gearing; such as directions for laying out blanks for cut gears, etc.

Grant's Teeth of Gears is a most excellent concise treatise on tooth outlines; and MacCord's Kinematics, which is devoted

mainly to gearing, is a very complete work, covering many points not ordinarily taken up and containing much original matter.

The discussion of this chapter has been very much abbreviated, as the subject is exhaustively treated in a large number of available books, and no attempt has been made to give more than a general treatment of fundamental principles.

CHAPTER V.

CAMS AND OTHER DIRECT-CONTACT MECHANISMS.

93. Cams.—The term cam is applied to a large and varied class of machine members which are often used to impart a more or less complex motion to a follower. Cams are most commonly employed for motions which cannot be easily produced by other simple forms of mechanism. A cam consists of a piece so shaped that its motion (which is usually a rotation, but often an oscillation or translation) imparts a definite, and ordinarily variable, motion to another member upon which it acts by direct contact, or through an auxiliary roll or block.

Figs. 36, 44, and 57 show common forms of cams. Such mechanisms as those illustrated in Figs. 38, 39, etc., and even gear-teeth, might be treated as special forms of cams: but it is more convenient to consider them by themselves.

There are two principal classes of cams: those with a curved edge or groove, which impart motion to a follower moving in the plane of the cam motion, and those which cause the driver to move in a different plane, usually perpendicular to the plane of the cam's motion. The latter class may be derived from the former, as will appear later. Figs. 140 to 147 represent cams of the first kind.

94. Cams moving the Follower in the Plane of the Cam by a Point or Roller.—Fig. 140 represents a simple case, in which the path of the follower is a straight line passing through the fixed centre about which the driver rotates. Suppose O to be the fixed centre of the cam, PM to be the path of a point P of the follower, and that P is to have positions corresponding to 0, 1, 2, 3, etc.; for

the angular motions of the driver o, α_1, α_2, etc. These angles may be equal or otherwise; and the follower may have a period of rest, or a "dwell," as indicated by the coincidence of the positions 4, 5. Lay off the radii $O1'$, $O2'$, $O3'$, etc., making the desired angles α_1, α_2, etc., with PO, and locate the corresponding positions of the follower, 0, 1, 2, 3, etc. With a centre at O and radius $O1$, draw an arc cutting $O1'$ at $1'$; with radius $O2$, cut $O2'$ at $2'$, etc. Through

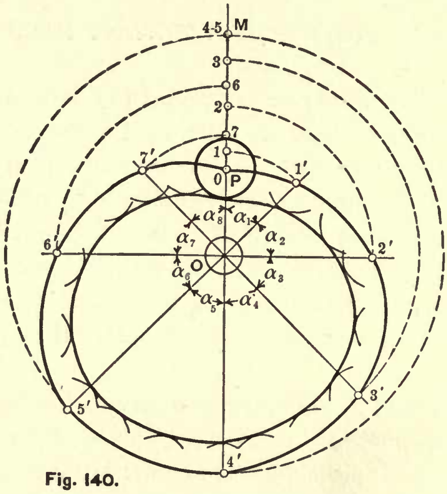

Fig. 140.

these intersections pass a smooth curve. Then this curve as it rotates will impart the required motion to the point P, which is supposed to be guided in the line PM, for when the cam has moved through the angle α_1, for example, the radius $O1'$ will coincide with the line PM, and $1'$ will be at 1. The same reasoning applies to all the points found above.

If the real cam is a solid (cylinder) of which the curve shown is one of the equal transverse sections, the follower would have to be a mere edge, to satisfy the conditions given. While such a combination satisfies the kinematic requirements, it would work with unnecessary friction and would wear rapidly; hence a derived form is used which gives the same motion to the follower.

To reduce the friction, a roller with its centre at P may be attached to the follower, as shown by the small circle in the figure. The profile of a cam to impart the required motion to the follower by acting on this roller is determined by the following construction: Take a radius equal to the radius of the roller, and with centres along the original outline draw arcs inside that outline. A smooth curve tangent to all these arcs is the required outline. The original outline is called the *pitch line* of the derived cam.

When the path of the point P of the follower is in a straight line which does not pass through O, or when it is curved, as in

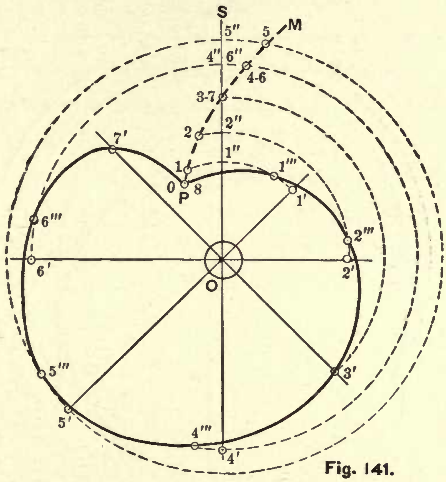

Fig. 141.

Fig. 141, the construction differs slightly from that of Fig. 140. The radii of the cam are laid off as before at angles corresponding to the required successive angular motions of the cam. When the radii $O1'$, $O2'$, etc., lie in the line OS, the follower centre, P, is not on these radii, but in the path PM, at 1, 2, etc. With O as a centre and a radius $O1$, draw the arc $1-1''-1'''-1'$. Now lay off on this arc, from its intersection with the radius $O1'$, a distance equal to $1-1''$, locating the point $1'''$. This is a point in the required pitch line. for when $1'$ is at $1''$, the point $1'''$ coincides

with 1. The other points in the pitch line are located in a similar way. The distances 1′–1‴, 2′–2‴, etc., are laid off to the right or left of 1′, 2′, etc., according as the positions 1, 2, etc., of the follower are to the right or left of OS.

The actual cam outline to act properly with a roll of any given diameter is obtained precisely as in the other example.

It is usually desirable to have the path of the follower as nearly in line with a radius of the cam as possible, as this condition gives less obliquity of action, especially with small cams.

95. Cams acting on a Tangential Follower to move it in the Plane of the Cam.—It is often desired to have the cam act tan-

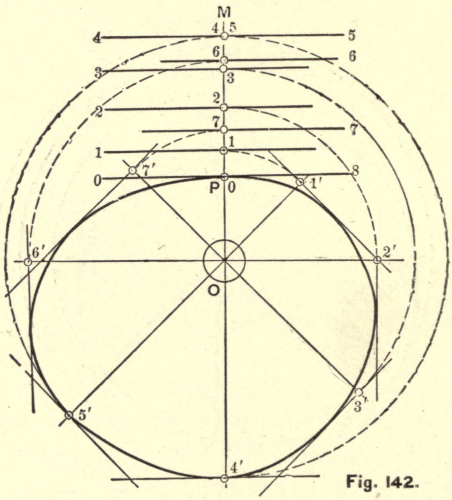

Fig. 142.

gentially upon a flat or a curved follower, as in Figs. 142, 143, or 144. In this case there are limitations to the motion which it is possible to impart to the follower. Thus in Figs. 142 and 143, in which the acting surface of the follower is a flat face (plane surface), it is evident that no portion of the acting surface of the cam can be concave, for such portions could not become tangent to the follower.

With the curved (convex) follower as shown in Fig. 144, it is

possible to have a concave portion of the driver, but this portion must have as great a radius of curvature as any part of the follower with which it acts. General methods of designing these tangential cams are shown in Figs. 142, 143, and 144.

In Fig. 142 the various positions of the follower are parallel to each other, and the acting face is preferably, but not necessarily, perpendicular to the direction of the follower's motion. Lay off radii of the cam, $O1'$, $O2'$, etc., marking desired angular motions from the original position corresponding to the given positions of the follower, 1–1, 2–2, etc. With a centre at O, draw arcs through the intersections of the various positions of the follower with the reference line OM, and cutting the corresponding radii of the cam, as shown, at $1'$, $2'$, etc. At $1'$, $2'$, $3'$, etc., draw lines making the same angle with the respective radii that the follower makes with PM. Draw a smooth curve tangent to all these lines last drawn.* This curve is the required cam outline; for when any radius, as $O\,3'$, lies on OM, $3'$ will lie at 3 and the tangent through $3'$ coincides with the required position of the follower.

If the successive positions of the follower are not parallel to each other, as in Fig. 143, the solution is as follows:

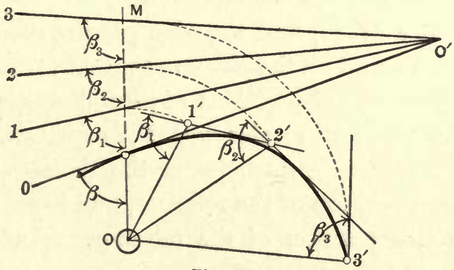

Fig. 143.

With a centre at O, draw arcs through the intersections of O'–1, O'–2, etc., with the reference line OM; cutting the respective

* If it is found impossible to draw a curve tangent to all these lines, a condition as to the successive positions of driver and follower has been imposed which cannot be met with this type of cam. If the curve has a sharp corner or crosses itself, the difficulty can usually be overcome by increasing the diameter of the cam.

radii $O1'$, $O2'$, etc., at $1'$, $2'$, etc. At $1'$, draw a line making an angle with the radius $O1'$ equal to β_1; at $2'$ draw a line making an angle with the radius $O2'$ equal to β_2, etc., then proceed as in the former example by drawing a curve tangent to these last lines.

When the follower is curved and has an angular motion as in Fig. 144, the following modification of the above method may be used. Connect O' with the points 1, 2, etc., where the successive positions of the curved edge of the follower cut OM. Draw circular arcs about O with radii $O1$, $O2$, etc., cutting the radii $O1'$, $O2'$, etc., which correspond to the desired angular motions of the cam. Draw a circle through O' with centre at O; then with radii $O'1$, $O'2$, etc., and centres at $1'$, $2'$, etc., cut this last circle in the points $1''$, $2''$, etc. Now form a templet with the curve of the follower for one edge, and a centre corresponding to O'. Place this centre at $1''$, with the edge of the templet passing through $1'$, and trace along the edge. In a similar way with the templet centre at $2''$, and the edge passing through $2'$, trace the curve; and so on for all the phases required. A smooth curve tangent to all these traced positions of the follower is the required cam outline; for when $2''$ rotates with the cam to O', $2'$ will fall at 2, and the corresponding tracing of the templet will coincide with the required position of the follower, and hence the cam will be tangent to this position of the follower. Similar reasoning applies to the other phases.

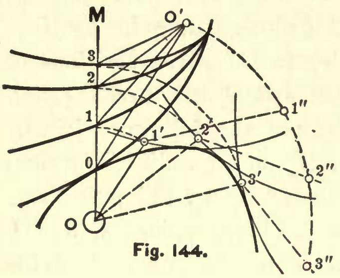

Fig. 144.

96. Positive Return of Follower.—In the forms of cams considered so far, no means has been shown for insuring a return of the follower after it reaches its position farthest from the centre of the cam. This is often accomplished through the action of gravity, a spring, or some other external force; but it is necessary under many conditions to completely constrain the motion of

the follower by the mechanism itself. The follower is then said to have a positive return. Positive return of the follower is sometimes obtained by providing a groove in which the roll of the follower acts as in Fig. 145. In this arrangement there are two defects, which may be serious, especially at high speeds. The slot must have a width somewhat in excess of the diameter of the roll, for both faces of the cam groove move in the same direction, and if the roll is in contact with the inner face of the cam it tends to rotate in one direction, while if it touches the outer face this tends to rotate the roll in the opposite direction. The figure shows the action when the inner face is acting on the roll. Owing to the necessary clearance there is some lost motion which is taken up when the follower reaches either of its extreme positions, and is acted upon by the other face of the cam. If the speed is high the taking up of this "slack" may result in a sharp blow or knock which makes a noise and may injure the mechanism. The other defect is due to the fact that when the roll changes contact from one face to the other, there is a tendency to instant reversal of its motion, but the inertia of the rapidly revolving roll resists this, resulting in a temporary grinding action which wears both cam and roll. Under slow speeds these actions may not be serious.

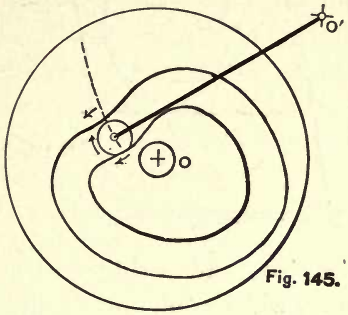

Fig. 145.

A method of returning the follower which overcomes these defects is shown in Fig. 146. Two rolls on opposite sides of the cam shaft are mounted as shown, or in some similar manner. A cam is designed, by the method given in Art. 94, to impart the required motion to one of these rolls. Every position of this roll causes the other to occupy a definite position, due to the connection between them, and a complimentary cam is designed

corresponding to these various positions of the second roll. This cam is placed beside the first one on the same shaft, and its action on the second roll keeps the first roll in contact with its cam, and imparts the return notion to the follower.

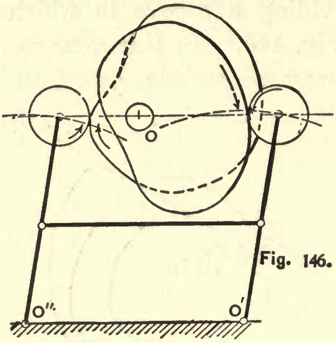

Fig. 146.

A single cam will give a positive return motion to a sliding follower having two rolls of equal diameter in contact with the cam on opposite sides of the centre. The rolls must be so mounted that a line joining their centres will pass through the centre of the shaft. With this construction the return motion of the follower will be an exact duplicate of the forward motion.

A complementary cam may also be used to secure positive return of the follower with cams of the types shown in Figs. 142, 143, and 144. When the forward and return motions are alike a yoke follower of the type shown in Fig. 147 may be used with a single cam.

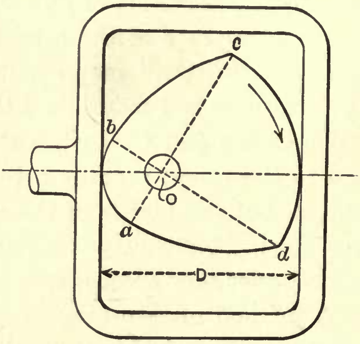

Fig. 147.

A cam like the one shown in Fig. 147 can be designed very easily, as it is bounded by circular arcs. The follower is shown in its extreme position to the right. There is a "dwell," or period of rest, at both extreme positions of the follower. The parts *ab* and *cd* are arcs with a common centre *O*, and the sum of their radii equals the distance between the parallel working faces of the follower, $=D$. To draw the arcs *bc* and *ad* take a radius equal to *D*, and draw an arc *bc* with a centre on *d*. This arc must be tangent to *ab* at *b*. Now with the same radius,

D, and a centre at *c* draw the arc *ad*, thus completing the outline. In the position shown the follower is at rest; when *a* comes in contact with the left-hand face of the follower, on the line of centres, *c* is in contact with the other face at a point directly opposite; then while *ad* acts upon the left-hand face of the yoke the follower moves to the left; this is followed by a period of rest while *dc* is in contact with the left-hand face, and then *ad* comes in contact with the right-hand face, returning the follower to the right. This cam, or a modification of it, is used to actuate the valves of the engines on the stern-wheel steamers of the upper Mississippi and its tributaries.

97. Translation Cams.—A form of cam which by its translation imparts motion to a follower is shown in Fig. 148. If it be required

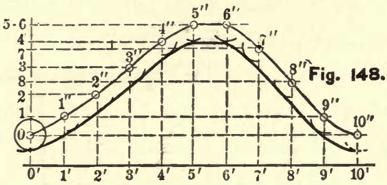

Fig. 148.

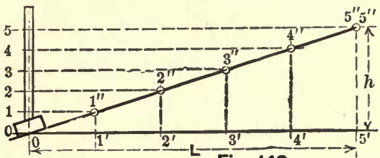

Fig. 149.

that a point of the follower shall be at the points 1, 2, 3, etc., as the points 1′, 2′, 3′, etc., of the driver coincide with 0′, design the cam as follows: Erect perpendiculars at 1′, 2′, etc., as 1′–1″, etc. Diaw a line from 1, parallel to the motion of the driver, cutting 1′–1″ in 1″; through 2 draw a parallel, cutting 2′–2″ in 2″, etc. A line through these successive intersections gives the pitch line of the cam. If 0, 1, 2, represent positions of the centre of a roll attached to the follower, the actual cam outline is formed as indicated in Fig. 140.

If the successive motions of the follower are to be proportional to those of the driver, the cam becomes an inclined plane, as shown in Fig. 149. In this case the flat shoe as shown may act directly on the cam, as it will fit the surface of the driver at all points. This will provide a larger contact surface and reduce wear, though it results in greater friction than when a roll is used.

98. Motion imparted to the Follower Perpendicular to Plane of Cam's Motion.—The cam of either Fig. 148 or 149 may be wrapped upon a right cylinder, as shown in Figs 150 and 151, and then the rotation of these cylinders about their axes will impart a motion to the follower parallel to these axes and exactly equivalent to the motion due to the translation of the original cams. The base lines, or lines parallel to the translation of the original cams, become circles when the cams are wrapped upon the cylinders.

If these cams are provided with grooves in which a roll acts, as in Fig. 150, clearance must be provided: for the opposite faces of the cam surface tend to rotate the roll in different directions; hence these cams are subject to the defects of the grooved cam men-

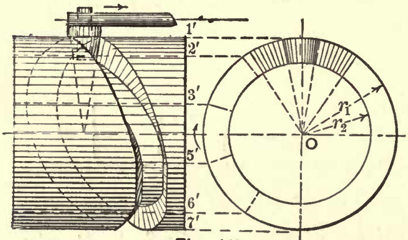

Fig. 150.

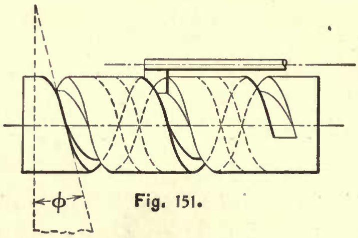

Fig. 151.

tioned in a preceding article. There is another peculiarity of the action of such a cam as that of Fig. 150, if the roll is a cylinder, which results in a grinding action between the cam face and the roll; but this is overcome by using a properly proportioned conical roll, as shown in the figure. If the outer radius of the cam is r_1 and the radius at the bottom of the working part of the groove is r_2, points at the outer edge have a linear velocity of $2\pi r_1 n$, while points at the inner portion have a smaller velocity, $2\pi r_2 n$. If the roll is a cylinder, the faces of the cam being perpendicular to the axis, all points on the contact element of this cylinder must evidently have equal linear velocities; hence the velocities of contact points of the driver and follower can only be the same at one point along the element of contact. If, however, the roll is the frustum of a cone with its apex at the axis of the cam and the

sides of the groove be given the corresponding slant, this difficulty is practically overcome. The shaded portion of the face in the section at the right of Fig. 150 shows the development of the acting surface of such a conical roll, or of a portion of such surface.

If the cam is to rotate continuously, instead of vibrating upon its axis, the original cam of Fig. 148 must have its first and last ordinates of equal lengths (measured from the base line, or from any line parallel to this); otherwise the groove would not be continuous when formed on the cylinder, and it could only drive the cam by reciprocation about its axis; when the return motion would be an exact reversal of the forward motion. The path of the follower may be other than a straight line, and the construction of the pitch line of such a cam is shown in Fig. 152. The points in the pitch line are on the parallels to the path of the driver; but to the right, or left, of the intersections of such parallels with the corresponding perpendiculars by distances equal to the given simultaneous distance of the follower from the reference line O–M. The construction of the cam from this pitch line is exactly similar to the cases already treated.

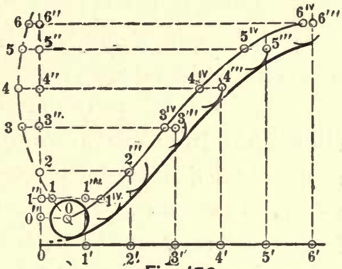

Fig. 152.

99. The Screw.—The cam of Fig. 151, as derived from the inclined plane of Fig. 149, will be recognized as the common screw, in which the sliding block of the follower corresponds to a portion of the thread of the nut.

The ordinary nut and screw differs essentially from this only in the length of the block of the follower, which is made to include several coils of the cam (threads of the screw) in order to distribute the pressure, reduce wear, and increase the strength. The groove in which the nut works may be rectangular, triangular, or of any one of many possible sections, without modifying the relative motion transmitted, which is governed entirely by the relation between the axial and circumferential components of the threads.

If the cam of Fig. 149 be wrapped upon a cylinder the circumference of which is $L = \pi D$, each revolution of the cam or screw will move the follower through a distance h, which is in this case the pitch of the helix or screw-thread. If this cam were wrapped upon a cylinder of half the above diameter, two revolutions of the screw would be required to move the follower the distance h, or the cam of length L, as shown, would make two complete coils of the helix, and the pitch of the screw (the distance from one thread to the next, parallel to the axis) would be $\frac{1}{2}h$. In any such case the inclination of the helix to the plane of motion of any point in it is the angle, ϕ, whose tangent is $h \div L$, and this inclination determines the velocity ratio of driver to follower, which is $L \div h$, at a contact point on the pitch line. Of course the groove has sensible depth in an actual screw, and then points on the screw-thread at different distances from the axis have different linear velocities relative to the nut. If the screw is driven by a crank or pulley of larger radius than the acting surface, the velocity of the actual point of application of the driving force is correspondingly greater, relative to the nut, than $L \div h$, but is still proportional to this quotient.

If the pitch of the thread is great enough to permit it, another similar thread may be cut between the grooves, as indicated by the dotted lines on Fig. 151, and corresponding extra threads of the nut may work in this groove. This does not alter the motion transmitted, but it gives more bearing surface for a given length of nut. Such a screw is called a double-threaded screw. There may be any number of such threads, if the proportions will permit, giving a multiple-threaded screw.

100. The Endless Screw. Worm-gearing.—Fig. 153 shows a screw with an angular thread and a small block (indicated by the shaded tooth) for a follower. Rotation of the screw in the direction indicated moves this follower to the left. If this block is pivoted at O', instead of being guided parallel to the axis of the screw, it will move in a circle about O' (provided it is relieved so as to avoid binding), and it soon passes out of action. Now if a series of such

pieces be arranged in a continuous circle about O', all connected together and properly spaced, as each piece passes out of action at the left another will engage at the right. If these blocks constitute a complete circular rim, the action will be continuous, and indefinite rotation of the driver will result in indefinite rotation of the follower. This arrangement constitutes, in a rudimentary manner, the common worm and wheel, or endless screw, as it is sometimes termed. If the teeth of the wheel in this figure were given a transverse section exactly fitting the spaces between the

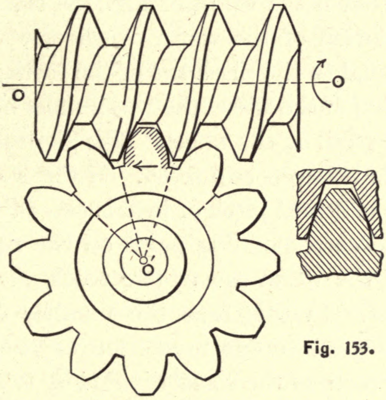

Fig. 153.

threads of the screw, as is the case in an ordinary nut and screw, the teeth would bind or interfere; but by giving a properly modified form to the teeth the action may be made smooth and uniform.

Suppose that the screw, or *worm*, as it is commonly called, be translated axially without rotation. It will then cause the wheel to rotate just as a pinion is rotated by a rack; and if sections of the teeth of the wheel and the screw-threads perpendicular to the axis of the wheel are correct forms for teeth of a pinion and rack, the velocity ratio will be constant. It will be apparent that the rotation of the screw is equivalent to this translation of the screw when acting as a rack; for by this rotation successive equal meridian sections of the worm are brought into action on the middle section of the wheel. It follows from this that these

sections should correspond to the forms proper for a rack and pinion. The sections of the teeth of an involute rack are trapezoids (sections of the acting faces being inclined straight lines perpendicular to the line of action, or the common normal). Then if the screw-threads of Fig. 153 have meridian sections bounded by inclined straight lines on the acting sides, the transverse section of the teeth should be corresponding involutes. The detail at the right of the figure indicates this form.

It is evident that if the worm of Fig. 153 is a single-threaded screw, each revolution of the worm causes the wheel to rotate through an arc equal to the pitch arc of the teeth; and to make a complete revolution of the wheel the worm must be given as many turns as there are teeth on the wheel. The pitch angle, as in spur-gears, is the angle between two teeth. If the worm is a double-threaded screw, the helical pitch or lead is twice the distance from one thread to the corresponding point on the next thread; and one revolution of the worm moves two teeth of the wheel past the line of centres. In this case the number of turns of the worm required to produce one revolution of the wheel is equal to the number of teeth of the wheel divided by two.

In general, the ratio of the angular velocity of the worm to that of the wheel equals the number of teeth of the wheel divided by the number of separate threads of the screw. The screw is called a worm only when it has but few threads. The number of threads may be increased indefinitely with a corresponding reduction of the velocity ratio. It is customary to design the teeth of the resulting gears in an entirely different manner, which has already been explained in the articles on helical gears.

In designing a worm and wheel, let $T=$ number of threads (teeth) on wheel; $t=$number of threads on worm; $p=$circular pitch of wheel$=$axial pitch of worm-threads; $D=$pitch diameter of wheel; $d=$pitch diameter of worm; $\Delta=$distance between axes;

ω and ω_1 = angular velocities of wheel and worm, respectively; ϕ = helix angle, or inclination of threads to transverse section of the worm.

$$\frac{\omega}{\omega_1}=\frac{t}{T}; \qquad \therefore\ T=\frac{\omega_1}{\omega}t.$$

This relation usually fixes T and t.

$Tp=\pi D$; $D=Tp\div\pi$; or, $p=\pi D\div T$. The strength of the gear depends upon p, and hence p should be fixed and D made to agree, if the conditions will permit. If, however, $\varDelta$ is fixed, D is limited, for $\frac{1}{2}(D+d)=\varDelta$; but D and d may have any value consistent with this requirement. The value of p gives the number of threads to the inch on the worm, and hence p, d and t give the helix angle ϕ; for $\tan\phi=pt\div\pi d$.

If the teeth of the wheel are ordinary screw-threads (all transverse sections of the wheel being identical in form) upon a cylindrical pitch surface, this pitch cylinder and that of the worm are tangent at a single point, and the teeth have point contact only. That is, the worm always engages with points on the central transverse section of the wheel. The worm-wheel may be made of the form shown in Fig. 154, when it is called a close-fitting wheel. The teeth of this wheel may be drawn by passing a series of planes through the worm, parallel to the axis and to each other and perpendicular to the axis of the wheel. Each of these sections of the worm will be a rack section, but they are not all alike. Then make the corresponding sections of the wheel those appropriate for wheels to work with such racks.* This process is tedious, and is seldom required in practice, as by the method of cutting the wheels it is not necessary to lay out the teeth. If a cast worm and wheel are to be made, it is of course necessary to lay out the teeth.

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* See Unwin's Machine Design, Part I, Art. 234, for a full description of this method of drawing worm-wheel teeth.

101. Hobbing Worm-wheels.—A worm-wheel may be accurately cut by the following process: Turn up the blank to correspond to the outside of the teeth (Fig. 154)*. Next cut a screw of tool steel to the exact form of the worm, then make a milling-cutter of this tool steel worm by cutting flutes across the threads, and " backing

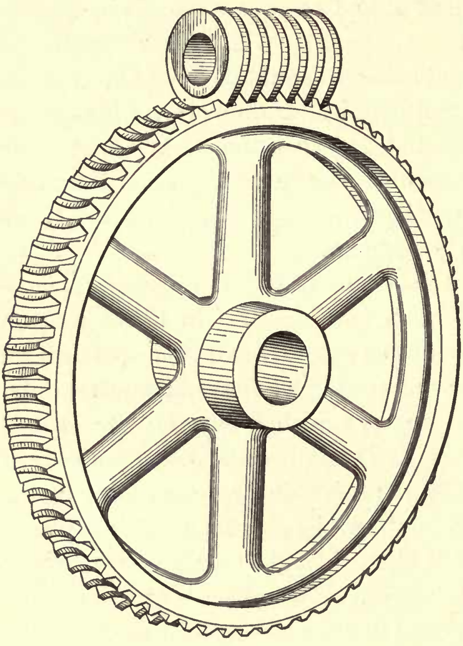

Fig. 154.

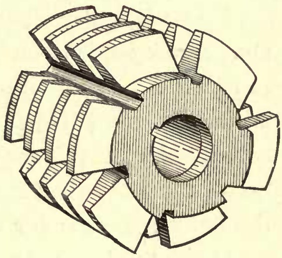

Fig. 154*a*.

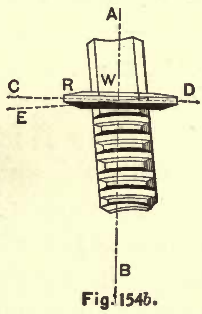

Fig. 154*b*.

off " the teeth thus formed for clearance. This is called a " hob " (see Fig. 154*a*), and it is hardened, tempered, and then used as a milling-cutter. The hob and worm-wheel blank are mounted in the gear-cutting machine, with their axes at right angles but necessarily somewhat farther apart than the desired distance

* Figs. 154, 154*a*, and 154*b* are taken from Brown & Sharpe's Treatise on Gears.

between the axes of the worm and wheel. They are then rotated about their axes with the velocity ratio that the worm and wheel are to have, and the blank is fed toward the hob very slowly until the distance between the axes is the same as the desired distance between the axes of worm and wheel. The wheel is sometimes caused to rotate simply by the driving action of the hob, the teeth of the wheel having been roughly cut or "gashed" with an ordinary milling cutter; but better results are attained when it is driven positively from the cutter-spindle, with the required velocity ratio, through a suitable train of gearing. It will be seen that the teeth thus formed on the wheel will work correctly with a worm which is an exact reproduction of the hob, except that the cutting-teeth are omitted.

The worm and hob may be cut like any screw in a lathe, with a tool which will give the desired form of threads.

Fig. 154*b* shows a method of cutting an approximate close-fitting worm-wheel with an ordinary gear-cutter, the diameter and section of which corresponds to the worm. If the cutter, as shown in this figure, is fed diagonally across the wheel-blank, a straight (point contact) wheel will be produced.

If the cutter is fed radially inward, toward the axis of the wheel, a "drop-cut" worm-wheel is produced. Such a drop-cut wheel resembles a hobbed wheel in form; but the method does not give a truly close-fitting wheel, such as is obtained by the hobbing process.

CHAPTER VI.

LINKWORK.

102. General Scope of Linkwork.—The simplest form of a constrained link-mechanism consists of four links, each pivoted at two points to adjacent links. A link with but two pivots, and joined to two adjacent members, is called a *simple link.* If a link has more than two such pivots and is joined directly by them to more than two separate members it is called a *compound link.*

A complete linkwork "chain," as link-mechanisms are sometimes called, cannot have less than four links; for if three links are connected in a closed chain they form a triangle, which is a rigid construction not permitting relative motion between the members. If more than four simple links are connected in a closed chain, forming a jointed polygon of more than four sides, a given motion of one member does not compel the others to move in a definite manner. Link-mechanisms of more than four members are used; but, in these cases, one or more of the members must be a compound link. Linkwork can be used to convert:

(*a*) Continuous rotation into reciprocation (rectilinear or circular) or the reverse.

(*b*) Reciprocation into reciprocation with a constant or a variable angular velocity ratio.

(*c*) Continuous rotation into continuous rotation, with a constant or a variable velocity ratio.

One or more of the links in a linkwork chain may be replaced by a sliding block or similar piece. Certain of these modified chains are of very great importance in practical machine construction.

103. The Four-link Chain.—The general form of the four-link chain is shown by Figs. 50 to 53 and other figures already given. It is now in order to examine the influence of the proportions of the members of the four-link chain upon the motion transmitted.

The following notation will be used: The driver will be designated as a; the follower, b; the connector, c; and the stationary link, or frame, d.

Fig. 155 shows a mechanism in which circular reciprocation of a produces circular reciprocation of b. The phase shown by the light lines a', b', c', is a limiting phase, and a can move no farther to the left (left-hand rotation).*

The driver (Fig. 155) can move through the arc a'–a–a''–a''', causing the follower to move from b' through b to b'', and then return over this path to b'''. Within these limits circular reciprocation can produce circular reciprocation, and either member might be the driver; but, practically, the action is not smooth when the

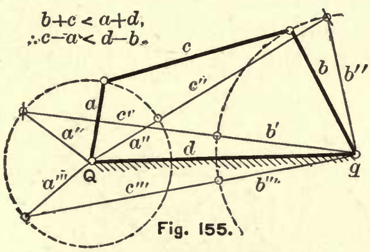

Fig. 155.

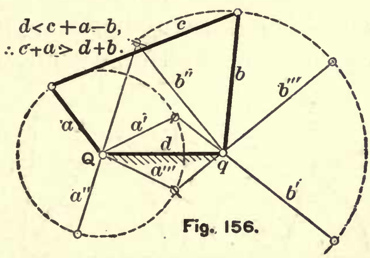

Fig. 156.

follower is near a dead-point; and if b is the follower, the range of action should be somewhat less than the maximum given. In this figure it will be seen that $b + c < a + d$; and it will be seen that

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* This position of the follower, b', is called a dead-point position; for there can be no component of the motion of the connector in the direction of its length, and hence no positive transmission of motion. When the connector and either the driver or follower lie in one straight line, a dead-point is reached. If the two members lie on opposite sides of their common pivot, as b' and c' in Fig. 155, the condition is called an outer dead-point position. If the links coincide in direction and are on the same side of their pivot, as b' and c' in Fig. 156, an inner dead-point is reached.

a cannot make a complete rotation unless $b + c$ is equal to, or greater than, $a + d$; or $c - a \geqq d - b$.

In Fig. 156 the follower reaches an inner dead-point when it is in the position b', and a can rotate no farther to the right than the position a'. In this case the driver can vibrate through the angle a'–a–a''–a''', causing the follower to reciprocate from b' through b to b'' and back to b'''. It is evident that $d <$ than $a + c - b$; and the driver cannot make a complete rotation unless d is equal to, or greater than, $a + c - b$; or $a + c \leqq d + b$. It will be noticed that the follower might pass (but cannot be positively driven by a) beyond the position b', in either Fig. 155 or Fig. 156; if this should occur in any way, the motion transmitted would be completely changed.

To sum up these two cases, we find that the driver cannot make a complete revolution unless these conditions are present: $c - a \geqq d - b$; and $c + a \leqq d + b$. If $c - a = d - b$, the driver and follower have simultaneously inner and outer dead-points, respectively, as shown by Fig. 157 (in the phase a', b'), and the

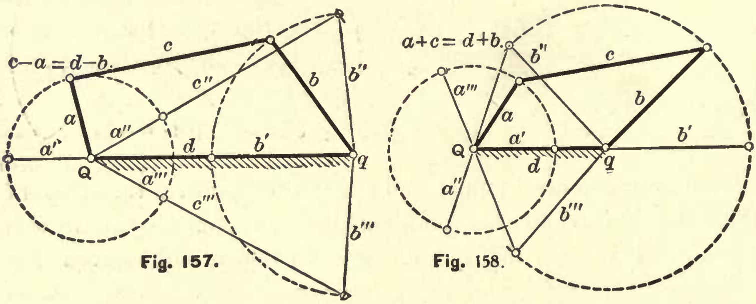


Fig. 157. Fig. 158.

motion of the follower may be towards either b'' or b''', as the driver passes this position. If $c + a = d + b$ (Fig. 158) the driver reaches the outer dead-point as the follower reaches the inner dead-point (a' and b', respectively); and the follower may either return to b'' or pass on to b'''. If $c - a > d - b$, and

$c + a < d + b$, as in Fig. 159, the motion of the follower is fully constrained, and the driver can make a complete rotation.

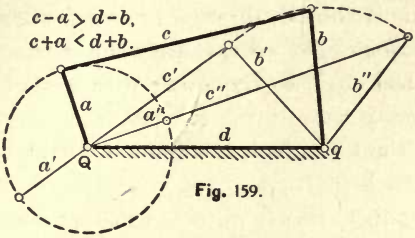


Fig. 159.

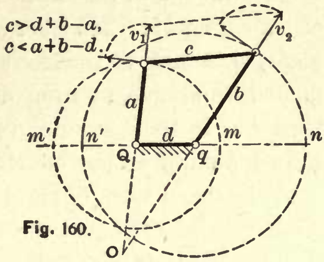


Fig. 160.

104. Continuous Rotation of both Driver and Follower.—(See Fig. 160.) If a single four-link chain is used to transmit positive continuous rotation to a follower from a rotating driver there must be no dead-points; for if the driver, a, reaches a dead-point, b will come to rest; and if b reaches a dead-point, its motion will not be fully constrained, and it will generally lock the driver, preventing complete rotation of the latter. With the proportions of Fig. 160, neither a nor b reaches a dead-point, and either of these members may be used as a driver, compelling the other to rotate continuously, but the velocity ratio will be variable. If rotation of both a and b is to be continuous, they will have simultaneous dead-points if *either* reaches a dead-point. If $c = mn = d + b - a$, a will have an outer dead-point, and b will have an inner dead-point at the same instant. If $c = mn' = a + b - d$, a and b will both have inner dead-points at one phase. Hence, for continuous positive rotation of both a and b the following conditions must be fulfilled: $c > d + b - a$, and $c < a + b - d$; hence, $d + b - a < a + b - d \therefore d < a$.*

The mechanism of Fig. 160 is called a "drag-link"; and it is sometimes used to connect the two arms of a centre-crank or double-throw crank. In this case a and b are equal, and d equals zero in the proper adjustment, that is, the fixed axes of a and b coincide.

* If dead-points are permissible (as in the parallel rods of locomotives), other provision is made for insuring that the dead-points shall be passed; then d may be, and usually is, greater than a.

As long as this condition is maintained, the link forms the equivalent of a rigid connection, and as the mechanism is reduced to a three-link chain, the motion transmitted is exactly similar to that of a solid crank. If either axis is shifted, through improper alignment, springing of the shaft supports, or wear, the motion is transmitted from one section of the shaft to the other with a slight variation in their angular velocity ratio during the revolution, and the wrenching action on the shaft is much less than it would be with the usual form of rigid crank-shaft.

If a and b are equal the angular velocity ratio is constant when d equals zero, or when $d = c$; for with these proportions the two perpendiculars from the fixed centres to the line of the link (c) are always equal for any phase. The former condition ($d = 0$) is that of the drag-link as applied to engine-cranks in proper alignment. The second condition ($d = c$) is one met in the locomotive side-rod connection; but in this case the driver and follower have simultaneous dead-points, and special means must be resorted to for complete constrainment of the follower.

The essentials of the locomotive side-rod connection are shown in Fig. 59, in which O and O' correspond to the centres of the connected wheels, $A'B'$ is the side rod (the dotted circles represent the paths of the pins by which the side rod is pivoted to the wheels); OA' and $O'B'$ (radii of the pin circles) are the driver and follower between which it is desired to transmit rotation with a constant velocity ratio; and OO' (the frame) is the fourth link. The full lines of Fig. 59 show a phase at which the driver and follower both lie on the line of centres. As the driver passes its dead-point position, the follower might move in either of the directions indicated by the arrows at B. Means of overcoming this defect in the constrainment will be shown later.

In the mechanism under consideration it is necessary that the four links shall form a parallelogram in all phases; that is, in Fig. 59, $A'B'$ must equal OO'; and OA' must equal $O'B'$. When this condition is fulfilled the angular velocity ratio must always be unity, for the perpendiculars from the fixed centres (O, O') to the con-

nector ($A'B'$) are equal in any phase (see Art. 30). In order to insure continuous rotation of the follower when the dead-points are passed, the simple mechanism of Fig. 59 must be supplemented. The method used on locomotives is shown in Fig. 161.

Each axle has two driving-wheels secured to it; the two wheels on either side being coupled by a side rod. The pins on the two

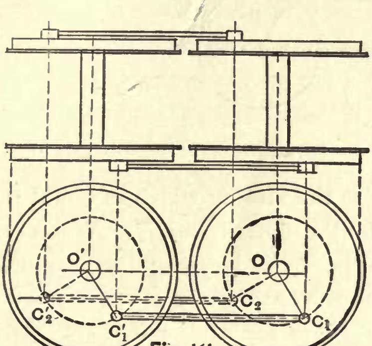


Fig. 161.

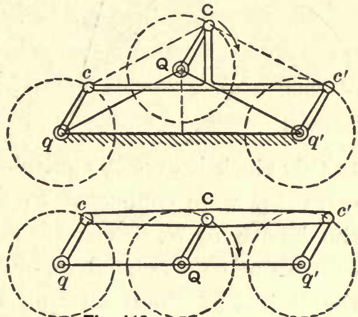


Fig. 162.

wheels of each axle are placed so that they are not in line, but one of these pins is ahead of the other, as shown in Fig. 161 by the angle $C_1OC_2 = C_1'O'C_2'$.

This angle is commonly 90°, so that when the system is at a dead centre phase on one side, the complementary system on the other side is in the best phase for transmission of motion.

Other possible arrangements to secure complete constrainment are shown in Fig. 162. In this case three equal cranks, not necessarily having their centres in one straight line, are connected by a rigid member (a compound link) which has a bearing for the pin of each. These bearings must be spaced to agree with the spacing of the fixed centres, and the cranks are always parallel to one another. The middle crank (shown with the arrow) should be the driver.

105. Combined Linkwork and Sliding-block Mechanisms.—The preceding articles of this chapter have been devoted to the four-link chain, and it was seen that by the mechanism of Fig. 159 a

circular reciprocation of the follower may be imparted by the reciprocation or rotation of the driver. Fig. 163 shows a mechanism in which one member of the four-link chain is replaced by a curved block b, sliding in a corresponding circular arc groove in an extension of the fixed link d. It is evident that the motion transmitted

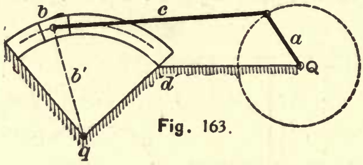


Fig. 163.

to this block is exactly equivalent to the motion which it would receive if it were connected to d by the dotted link b'. Whatever the length of b', it may be replaced by a block and groove, the centre line of which corresponds to the path of the moving end of the link b', without altering the character of the motion. Any other link might be replaced in a similar way by the equivalent slot and block. When any one of the links is very long this substitution of a sliding-block for a link may be convenient, the radius of curvature of the slot being equal to the length of the link replaced. If the link is of infinite length the centre line of the slot becomes a straight line, and the motion of the block is then a rectilinear translation.

The mechanisms shown in Figs. 69, 70, 71, and 72 represent this modified form of the four-link chain, or what Reuleaux has called the "slider-crank chain." This mechanism is so prominent in practical machine construction that it will be treated in detail.

106. Crank and Connecting-rod.—The connection between the piston and crank of the ordinary direct-acting engine, as shown in Fig. 27, is one of the most important examples of the modified four-link chain. In this case the motion of the piston and cross-head relative to the frame is equivalent to that of a link of infinite length. The piston and cross-head, being rigidly connected, are kinematically one piece; though we may be only concerned with

the motion of the piston, it is often convenient to speak of the motion of the cross-head, which is identical with that of the piston.

With the usual arrangement the path of the cross-head is in a line which passes through the crank-centre, and this line will be spoken of as the centre line. This is not a necessary condition, and it is sometimes departed from with a result that will be discussed in a later article. Unless otherwise stated it is to be understood that this ordinary arrangement is meant.

In the steam-engine the reciprocating piston is the driver and the crank is the follower. In case of an air-compressor or power-pump, the reverse is the case; but the relative motion of the members is not affected by this relation, as it depends simply upon the proportions of the mechanism.

The crank usually has a uniform rotation, or approximately such, and this condition will be assumed in the following discussion. It is evident that the cross-head (piston) must come to rest as the crank-pin passes the line of centres (dead-centre positions). Its motion is accelerated as it leaves either extreme position, attaining a maximum velocity near the middle of the stroke, followed by retardation (negative acceleration) through the latter part of the stroke. The motion of the reciprocating parts is approximately harmonic, departing from true harmonic motion more as the ratio of connecting-rod length to crank length becomes smaller.

The position of the crosshead, c (Fig. 164), for any crank posi-

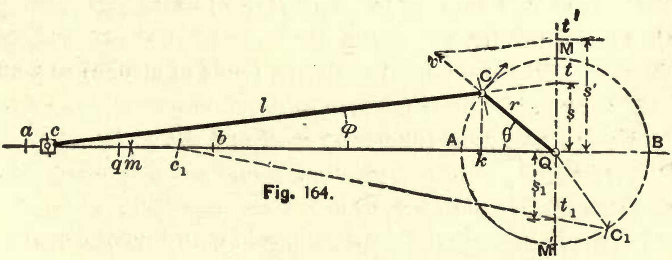


Fig. 164.

tion C, is obtained graphically by taking a radius equal to the length of the connecting-rod Cc, and, with a centre at the given crank position, cutting the path of the cross-head by an arc of this

radius. In a similar way the crank position corresponding to any cross-head position is found by taking a centre at the given cross-head position and cutting the crank-circle with an arc of the above radius. This last process gives two intersections, one above and one below the line of centres, as it should; for the cross-head passes the same point in its path during the forward and return strokes, both of which are accomplished during a single revolution of the crank. If a series of equidistant cross-head positions are taken, it is evident that the corresponding crank-pin positions will not be equidistant, and *vice versa*. That is, equal increments of cross-head (piston) motion do not impart equal increments of motion to the crank.

In drafting-room practice these graphic methods of finding simultaneous crank and cross-head positions are usually most convenient; but sometimes it is desirable to use analytical expressions for the relations between the crank and connecting-rod.

The more important kinematic relations will be derived, trigonometrically, using the following notation (see Fig. 164):

Centre of crank-circle $= Q$.
Centre of crank-pin $= C$.
Centre of cross-head pin $= c$.
Length of connecting-rod $= l$.
Length of crank $= r$.
Ratio of connecting-rod to crank $(l \div r) = n$.
Crank dead-centres $= A$ and B.
Corresponding cross-head positions (ends of stroke) $= a$ and b.
Mid-stroke cross-head position $= q$.
Mid-crank positions (quarters) $= M$ and M'.
Simultaneous cross-head position $= m$.
Crank angle, ahead of A, $= \theta$.
Corresponding angle of connecting-rod with line of centres $= \phi$.

Drop a perpendicular, Ck, from C upon the centre line; then $Ck = r \sin \theta = l \sin \phi$; $Qk = r \cos \theta$; $ck = \sqrt{l^2 - (Ck)^2} = \sqrt{l^2 - r^2 \sin^2 \theta}$.

For any crank angle, θ, the distance from c to $Q = ck + Qk = \sqrt{l^2 - r^2 \sin^2 \theta} + r \cos \theta$.

When C is at the quarter (M or M'), c is at m, a distance from its mid-position $= mq = Qq - Qm = l - \sqrt{l^2 - r^2} = nr - r\sqrt{n^2 - 1} = r(n - \sqrt{n^2 - 1})$, a quantity which increases as n decreases, and equals zero when $n =$ infinity.

It is seen from the above expression that when the crank has rotated through 90° from A, the cross-head has moved through more than half its stroke; while for the next 90° crank rotation the cross-head moves through less than half its stroke. It follows that, with uniform rotation of the crank, the half-stroke, aq, is made in less time than the half-stroke, qb, this variation decreasing as the connecting-rod length increases. The influence of this angularity of the rod on steam distribution will be seen to be important, when the subject of valve motions is studied.

In the illustrations of velocity diagrams (see Art. 41 and Figs. 69 and 76) it was shown that the ratio of the linear velocities of the crank-pin and cross-head is equal to the ratio between the length of the crank and that segment of a perpendicular to the line of centres through the shaft which lies between the centre line and the line of the connecting-rod, the latter prolonged if necessary. Thus, in Fig. 164, if QC represents the velocity of the crank-pin, to some scale, $s = Qt$ is the velocity of the piston (to the same scale) when the crank is at C. If the crank-pin velocity can not be represented conveniently to this scale, lay off Qv, along the line of the crank, to represent its velocity, and draw vt' parallel to cC; then $Qt' = s'$ is the required velocity of the piston to the scale assumed; and the value thus obtained for the piston velocity can be used, as in Fig. 76, for constructing the velocity diagram.

Another method of determining the ordinates of the velocity diagram is shown in Fig. 165. With this method, Cv is laid off on the extension of the line of the crank to represent the crank-pin velocity to a convenient scale; an ordinate is erected at c, and this is cut by drawing the line vv' parallel to the connecting-rod; then

cv' gives the velocity of the cross-head for this phase (Art. 40). The linear velocities of C to c are in the ratio of OC to Oc; as C and c are two points in the connecting-rod, which at the instant has a motion equivalent to a rotation about the instant centre O; hence the velocity of C is to that of c as OC is to Oc. The construction of the complete diagram will readily be seen from the figure. The method is the same as that used for the four-link chain in Fig. 77.

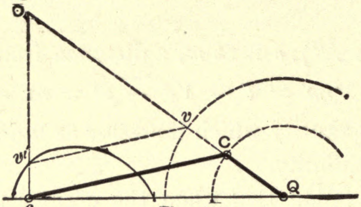


Fig. 165.

From Fig. 164 it will be seen that the velocity of the piston is equal to that of the crank-pin when $s = r$. There are two positions of the piston in each stroke where this condition is fulfilled. The first of these positions is shown in Fig. 166, where cC, produced, passes through M; hence $s = QM = r$. This equality of velocities can be seen directly by locating the instant centre O; for OCc is similar to QCM at this phase; hence $OC = Oc$, and the linear velocity of $c =$ linear velocity of C. The same relation can be shown by resolution of the velocities.

Fig. 166.

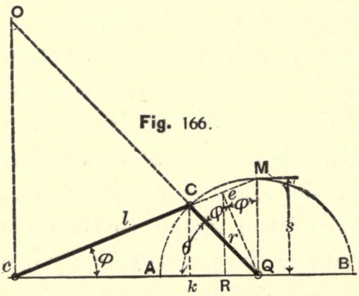
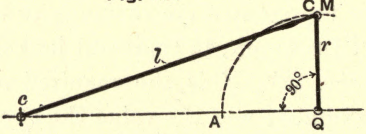


Fig. 167.



When C (Fig. 167) coincides with M, $s = QC = QM$, and the crank-pin and the piston have the same velocity. Between these two positions of equal crank-pin and piston velocity, the piston moves *faster* than the crank-pin; for s is greater than r. (See Fig. 169.)

The second of these positions of equal velocity is *always* that

at which the crank is perpendicular to the line of centres, and is independent of the ratio of connecting-rod to crank; provided this ratio is greater than unity. The first position is a function of this ratio; and the crank-angle corresponding to this phase is found as follows (Fig. 166): Let fall Qe perpendicular to CM, then as QMC is isosceles, QMe and QCe are equal triangles, and the angle $eQC = eQM = \phi$, $\therefore\ MQC = 2\phi$. $\therefore\ \theta = 90 - 2\phi$. $Ck = l \sin\phi = r\sin\theta = r\sin(90-2\phi) = r\cos 2\phi$, $\therefore\ \frac{l}{r}\sin\phi = n\sin\phi = \cos 2\phi; = 1 - 2\sin^2\phi$, $\therefore\ 2\sin^2\phi + n\sin\phi = 1$; dividing by 2 and completing the square:

$$\sin^2\phi + \frac{n}{2}\sin\phi + \frac{n^2}{16} = \tfrac{1}{2} + \frac{n^2}{16}$$

$$\therefore\ \sin\phi + \frac{n}{4} = \pm\sqrt{\tfrac{1}{2} + \frac{n^2}{16}} = \pm\tfrac{1}{4}\sqrt{8 + n^2},$$

$$\text{and } \sin\phi = \pm\tfrac{1}{4}\sqrt{8 + n^2} - \frac{n}{4}.$$

The double sign of the radical may be dropped, for if the minus sign be taken, with any value of n greater than 1, we would get a value for $\sin\phi$ *numerically* greater than 1, which is impossible. Taking the plus sign:

$$\sin\phi = \tfrac{1}{4}\left\{\sqrt{(8 + n^2)} - n\right\}$$

As $\sin\theta = n\sin\phi$,

$$\sin\theta = \frac{n}{4}\left\{\sqrt{(8 + n^2)} - n\right\} = \frac{n}{4}\left[\sqrt{(2.828)^2 + n^2} - n\right].$$

This form is convenient for graphical solution as follows (Fig. 168): Lay off distance $AB = n$ to a scale of $r = 1$, and erect a perpendicular $BC = 2.828$ to the same scale. Connect A and C, then the hypothenuse $AC = \sqrt{(2.828)^2 + n^2}$. With A as a centre and AB ($= n$) as a radius, describe arc BD, cutting AC in D.

$$DC = \sqrt{(2.828)^2 + n^2} - n.$$

$$\therefore\ DC \times \frac{n}{4} = \sin\theta;\ \text{or } DC \times \frac{l}{4} = r\sin\theta = Ck. \quad \text{(Fig. 166.)}$$

Between the two positions of the crank at which the crank-pin velocity equals the velocity of the reciprocating parts, the velocity of the latter is greater than that of the crank, as noted above. These

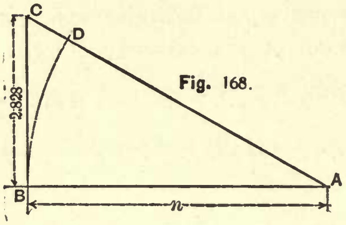


Fig. 168.

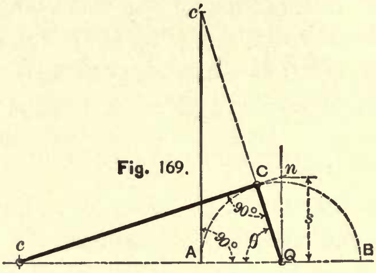


Fig. 169.

reciprocating parts (piston, piston-rod and cross-head) have very nearly the maximum velocity at the position where the connecting rod and crank form a right angle at C (Fig. 169). The true phase for the maximum velocity of the piston is a little later than the above position; but it is difficult to locate this exact position, and with the proportions of crank and connecting-rod used in ordinary engines ($l \div r = n =$ from 4 to 6 usually), this error is of no practical account, and the approximation is much more conveniently used. To find the crank position (Fig. 169) at which the crank is perpendicular to connecting-rod, erect Ac' perpendicular to the line of centres at A, equal to the length of the rod, and connect c' with Q. The intersection of $c'Q$ with the crank-circle locates the required position of the crank, C; for $Ac' = Cc$; $AQ = CQ$; and in the two triangles AQc' and CQc, the angle $AQC = \theta$ is common. When two sides and the corresponding angle of two triangles are equal the triangles are equal; therefore, as QAc' is a right angle by construction, cCQ is also a right angle, and C is the position of the crank-pin required. The phase at which the piston has its maximum velocity is of importance in certain problems relating to the mechanics of the steam-engine, for it is the phase at which the acceleration of the reciprocating parts is zero. In high-speed engines the acceleration of the reciprocating parts has a very important bearing upon pressures transmitted from the piston to the crank.

107. The Eccentric.—The eccentric is a modified crank, and all that has been said in the preceding article applies to the eccentric and rod. If the crank-pin be gradually enlarged, its throw remaining unchanged, the motion transmitted to a given connecting-rod is unaltered. Fig. 170 shows such a crank-pin enlarged in diameter until it includes the shaft, and it gives the familiar eccentric and rod.

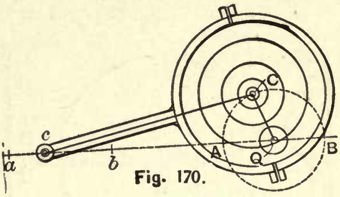


Fig. 170.

The throw of the eccentric is the radius of the equivalent crank, QC; or it equals the distance from the centre of the eccentric to the centre of the shaft about which it turns.

The enlargement of the pin increases the friction, although it has no kinematic effect. The eccentric is a useful expedient when a crank of small throw is required which cannot be conveniently located at the end of the shaft, for under such conditions the ordinary connecting-rod would "interfere" with the shaft unless a double-throw crank were used, and this latter form would weaken the shaft by cutting into it, besides being a more expensive construction. For these reasons the eccentric is very commonly employed for operating the valves of engines, imparting a reciprocating, and nearly harmonic, motion to them.

108. Connecting-rod of Infinite Length.—It has been seen that the stroke of the cross-head (Fig. 164) equals the diameter of the crank-pin circle, $= 2r$; and that the obliquity of the connecting-rod distorts the cross-head motion from a true harmonic motion, causing the half-stroke farthest from the shaft (at the head end of the cylinder) to be made in less time than is taken by the half-stroke nearest the shaft (the crank end). It was shown in Art. 106 that the displacement of the piston from mid-stroke, when the crank is at either "quarter," or $\theta = 90°$ (measured, in Fig. 164, by qm) is less as the connecting-rod is made longer, relative to the crank; or as $l \div r = n$ becomes greater.

If the rod were of infinite length, the cross-head would be at the middle of its stroke when the crank is at the quarter ($\theta = 90°$);

for it was shown that $mq = l - \sqrt{l^2 - r^2}$; hence, $mq = l - l = o$, when the length of the rod is infinity. It is, of course, impossible to have a rod of infinite length; but it was shown in Art. 105 that the cross-head and guides give the equivalent of an infinite length of link as to one member of the four-link chain; and the slotted rod and block of Fig. 171 may be introduced as an equivalent to an infinite connecting-rod. That is, this mechanism is the equivalent of the four-link chain with *two* links of infinite length.

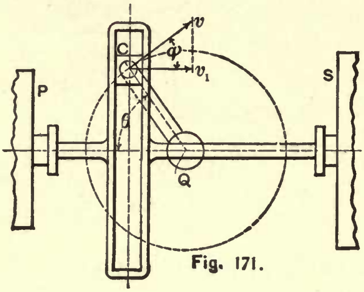


Fig. 171.

With the mechanism of Fig. 171 the crank is acted upon by the slotted rod through the block. The component of the motion of the crank-pin, which is normal to the acting faces of the yoke, equals the motion of the rod. This normal component is seen to equal the motion of the crank-pin multiplied by cos ϕ; and as $\phi = 90° - \theta$, cos ϕ = sin θ, hence the velocity of a piston attached to the slotted rod is equal to $v \sin \theta$, when v is the velocity of the crank. The piston velocity is a maximum when sin θ is a maximum (assuming the crank to rotate uniformly); or when $\theta = 90°$. At this phase the piston and crank have the same velocity, since sin $\theta = 1$. This agrees with the statement in Art. 106 that the velocity of the piston always equals that of the crank when $\theta = 90°$. With the finite rod there is another crank position, for a smaller value of θ, at which this equality also exists, and between these two crank positions the piston velocity is greater than that of the crank. As the rod is increased in length, these two positions for equality approach each other, the first one more nearly corresponding to $\theta = 90°$. With the infinite rod the two phases for equality coincide, and the phase for maximum velocity, which in the general case lies between them, also falls at $\theta = 90°$, as seen above. These conditions will be found to harmonize with the general relations deduced above. Fig. 171 indicates the application of this mechanism, as it is some-

times made to steam fire-engines and other steam-pumps. P indicates the pump-cylinder and S the steam-cylinder. The crank-shaft carries the fly-wheel.

The practical effect of this "rod of infinite length," or the Scotch yoke, as it is frequently called, is to make a more compact mechanism than would be obtained with a finite rod of ordinary length; for the distance between the "glands" of the stuffing-boxes on the two cylinders needs be only equal to the stroke plus the outside width of the slotted yoke, with a small allowance each side for clearance.

The sliding-block is not an essential, kinematically, as the crank-pin could act directly on the faces of the slot; but, as shown in Art. 28, it is generally desirable, when the conditions will permit, to use surface contact instead of line contact, thus distributing the pressure transmitted over a larger area.

The sliding of the block in the slotted member produces friction and resultant wear, which is not so easily overcome as in a pin connection; and the ordinary form of connecting-rod is therefore preferred as an engine connection when the utmost compactness is not a leading consideration.

109. Connecting-rod of Length Equal to Crank.—If the connecting-rod is of a length equal to the throw of the crank, as in Fig. 172, these two members always form an isosceles triangle, with the intercept on the centre line between the cross-head and shaft as a base. The distance $Qa = r + l = 2r$, and b, the end of the stroke next to the shaft, coincides with Q. In this arrangement, c would be drawn from a to Q during a crank movement AM and the displacement from the centre of stroke, due to angularity of the rod, $= AQ = r$. If the cross-head comes to rest at Q when C reaches M, with any farther motion of the crank the con-

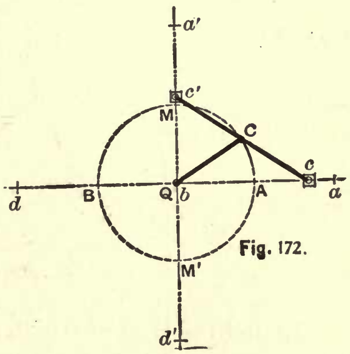


Fig. 172.

necting-rod would simply rotate around Q with the crank. If the cross-head continues to move in the line aQ, produced beyond Q, the crank movement MB would drive the cross-head to d, a distance from $Q = 2r$, and the total stroke of the cross-head would $= 4r$. The inertia of the cross-head as it approaches Q would tend to produce this effect; but such a motion can be made positive by the mechanism shown by the extension of cC to c'. In this form the rigid rod cc' $(= 2r)$ is pivoted at its centre to the crank-pin, and cross-heads at the ends of the rod move in guides at right angles to each other which intersect in Q. When C is at A, c is at a, and c' is at Q. As C moves to M, c moves to Q, and c' moves to a'; then, as the motion of C continues to the position B, c passes Q, moving to d, and c' returns to Q. As C passes B and moves to M', c' passes from Q to d', and c returns to Q. During the completion of the crank revolution, C moves from M' to A, c moves from Q to a, and c' returns to Q, completing the cycle.

At any phase the distance of c' from Q corresponds to Qt, of Fig. 164, and hence is proportional to the velocity of c; likewise, Qc is proportional to the velocity of c' at any phase. In this mechanism there is a transverse stress, as well as tension or compression on the rod cc'.

110. Path of Cross-head Passing Outside of Shaft-centre.—If the line of cross-head motion, gh (Fig. 173), does not pass through Q, the motion is modified as follows:

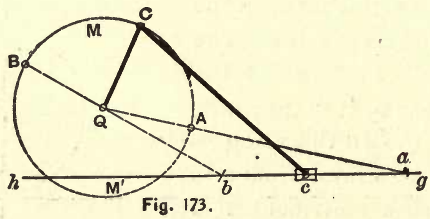


Fig. 173.

To find the ends of stroke a and b: first, take a radius $= l + r$, with a centre at Q, and cut gh in a; second take a radius $= l - r$, with the same centre, and cut gh in b; the required points are a

and b, and the corresponding crank positions are QA and QB. The stroke from a to b is made while the crank moves through the arc AMB; and the return stroke takes place as C moves through the arc $BM'A$. If the crank motion is uniform, the forward and return strokes are made in unequal times, and this mechanism gives one form of "quick-return motion." If it is required to design such a quick-return motion, the relative times of forward and return strokes being given: draw the crank circle and divide its circumference into two arcs having the required ratio, AMB, $BM'A$. Extend the radii through these points of division A and B, in directions QA and BQ; then lay off from Q on the extension of QA, $l + r$, locating a; and on BQ lay off $l - r$ from Q, giving b; a and b are the ends of the stroke, and ab is the line of cross-head motion.

In the Westinghouse engine the above construction is applied; that is, the line of piston travel passes to one side of the shaft-centre. Two cylinders are placed side by side, with connecting-rods acting on cranks which are opposite each other (180° apart). This engine is single-acting, steam acting on each piston only during its downward stroke; therefore, by giving the quick return to the upward stroke, one piston makes its exhaust-stroke and takes steam again before the other piston has quite completed its "working" stroke; thus, there is no period at which the rotative effort is absolutely zero. Furthermore, the greatest angularity of the connecting-rod occurs on the exhaust-stroke, and for a given length of connecting-rod, the maximum obliquity of action is reduced for the stroke during which steam-pressure is acting on the piston. Or, to state the case somewhat differently, the length of the rod can be reduced for a given maximum obliquity during the period of heavy pressure, thus permitting a more compact construction.

111. Motion of a Point in the Connecting-rod between the Cross-head and Crank.—In certain valve-mechanisms, motion is taken from some point in a connecting-rod (or eccentric rod) other than either of the pin-centres previously considered. Let P (Fig. 174) be such a point, the motion of which it is desired to find.

Find the instant centre O for any chosen phase of the rod Cc. All points of the rod, at the instant, rotate about O with the same angular velocity, and with linear velocities proportional to their radii. Hence, the linear velocity of P is to that of C as OP is to OC. The direction of the motion of P is perpendicular to OP, as indicated by Pv_3.

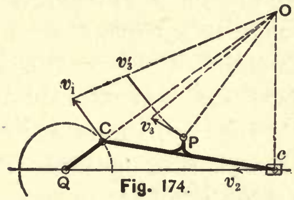


Fig. 174.

A similar method can be used if the point P lies beyond either the crank or cross-head in an extension of the connecting-rod.

This problem can be solved by the resolution and composition of relative velocities also, but not so readily.

112. Inversion of Crank and Connecting-rod Chain.—It was shown in Art. 39 that a kinematic chain may have the appearance of entirely different mechanisms when different members of it are held stationary. Thus, Figs. 69, 70, 71, and 72 show the four possible inversions of the crank and connecting-rod chain. The case of Fig. 69 has been treated in preceding articles of this chapter. Fig. 70 represents the condition when the former crank is made the fixed member; this case is next in practical importance to the ordinary crank and connecting-rod mechanism. This form may be used to secure a variable angular velocity of a continuously rotating follower from a uniformly rotating driver. It somewhat resembles the drag-link in its action. In conjunction with another linkage this mechanism is frequently used to produce a slow advance and a quick return of the cutter-bar of a shaping-machine.

The condition shown in Fig. 71 is, as already pointed out, the mechanism of the oscillating steam-engine. The case of Fig. 72 has comparatively little practical application. Any of these can be readily analyzed by the instant centre method. The form in which a is fixed (Fig. 70), will be treated in some detail, on account of its extended practical use; the others will not be taken up as special forms.

Fig. 175 shows a crank a which rotates about O and is pivoted to a sliding block by the pin P. This block fits a slot in the arm

b, which rotates about O'. The stationary member d supports the fixed centres O and O'. The point P rotates in the circle AB; hence, its motion at any instant is perpendicular to the radius PO (the centre line of the crank a). The velocity, which is usually uniform but not necessarily so, is designated by v_1. We may consider the point P to act upon the centre line (pitch line) of the slotted member, as the block does not affect the kinematic problem.

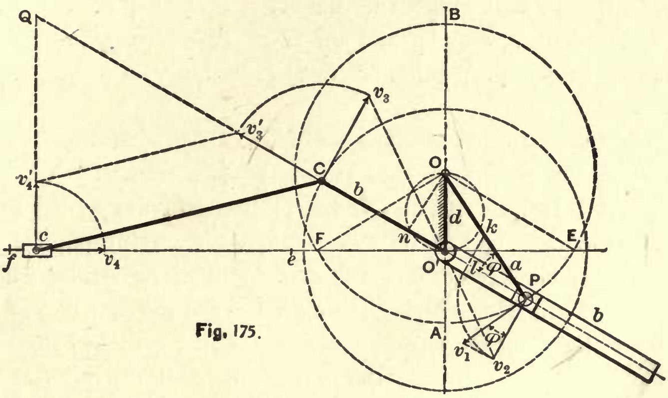


Fig. 175.

The point in a which lies at P has the velocity Pv_1, and the point in the slotted bar b, which is also at P for the instant, has the velocity Pv_2. As these two velocities have equal components along the common normal to the contact surfaces, the normal component of $Pv_1 = Pv_2$. As a point in a, P is fixed at the distance OP from the centre O. As a point in b, it travels back and forth along the pitch line of the slot, its distance from O', or its effective radius, varying from $O'A$ to $O'B$ as the driver moves from A to B. During the next half-revolution of the driver (B to A) the effective radius of the follower decreases from $O'B$ to $O'A$, thus completing the cycle.

Only the component of the motion of the driving-point which is normal to $O'P$ can impart rotation to the follower. The velocity of this component is represented by $v_2 = v_1 \cos \phi$ (in which expression ϕ is equal to the angle OPO'); because v_1 and v_2 are perpendicular to

OP and $O'P$, respectively. If a circle be drawn on OO' as a diameter, the intercept, Pn, of the centre line of the follower (extended through O' if necessary), which lies between P and this circle is to v_2 as the constant radius of the driver is to v_1. Or, $: v_1 : v_2 :: PO : Pn$; for, as OnO' is an angle subtended in a semicircle, On is perpendicular to $O'P$, hence $Pn = PO \cos \phi$. The velocity of the driver may be represented to a scale which will make it equal to PO, when Pn becomes the velocity v_2. If this velocity scale is not convenient, v_1 may be laid off from P towards O, as Pk, and a line kl drawn perpendicular to $O'P$ will give $Pl = v_2$, to this latter scale.

113. Quick-return Motions.—If (Fig. 175) a sliding block, c, travels in the path ef, which passes through O', and is connected to a point C in an extension of the slotted follower by the rod Cc, it will reach one end of its stroke when the driving-point P is at E, and this block c reaches the other end of its stroke when P is at F. While P is moving through the arc FBE, c moves from e to f; while P moves through the arc EAF, c makes its return stroke from f to e. Now if the driver rotates uniformly the times of these forward and return strokes are in the ratio of the arcs FBE to EAF. This is, in principle, the Whitworth quick-return mechanism, as it is frequently applied to shapers. The slow stroke is used for the cutting stroke of the tool, while the return stroke is made more rapidly, thus economizing time and increasing the capacity of the machine.

In designing such a mechanism the circle in which P rotates may be drawn with O as a centre; then divide its circumference by E and F into two arcs having the ratio desired for the times of the forward and return strokes. Draw a line through EF, extended to one side, and the path of c lies in this line. Drop a perpendicular from O upon EF and its foot will locate O', the fixed centre for the slotted arm. Take C at a distance from O', which will give the required length of stroke, and choose a suitable length for the connecting-rod Cc.

In practice C is a pin which can be set at different distances

along a radius to O', so that the length of stroke of c can be varied to suit the work. The pin C might be placed on the same side of O' as the slot; but it is usually more convenient to locate it as in Fig. 175.

The velocity of C is to v_4 as $O'C$ is to $O'P$, since these are the velocities of two points in one piece which rotates about O'. The motion of C is perpendicular to $O'C$, as shown by Cv_3. To find its velocity lay off v_2 (found as above) and draw the line $v_2O'v_3$ cutting the perpendicular to $O'C$ at v_3, and giving Cv_3 as the velocity sought. To find the velocity of c, erect at c a perpendicular to its path; lay off $Cv_3' = Cv_3$ on the extension of $O'C$, and draw a line $v_3'v_4'$ parallel to the rod Cc; v_4' is an ordinate of the velocity diagram of c. The student should complete this diagram for both strokes, by the method indicated.

When the location of the instant centre O_{ac} can be determined, the linear velocity of c corresponding to the given linear velocity of P may be determined directly by the general method outlined at the end of Art. 40.

The practical construction of the Whitworth quick-return motion is shown in Fig. 176, in which the letters correspond to those of Fig. 175. The pin P is attached to a gear which rotates about O,

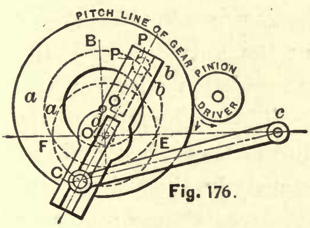


Fig. 176.

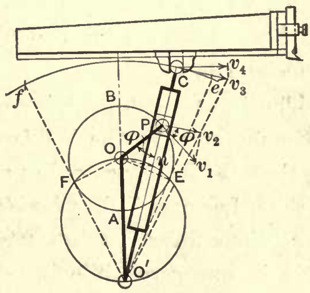


Fig. 177.

the centre of a large fixed stud. The centre O' is a pin secured in the fixed stud, and the slotted member rotates about this centre O'. The pin C can be clamped at different points along its slot to secure corresponding lengths of stroke of c.

Another quick-return mechanism, also much used for shapers, is indicated by Fig. 177. The slotted bar is pivoted to the frame at O', and is driven by the crank pin P, which rotates about O as in the preceding case. The slotted bar vibrates between the positions $O'e$ and $O'f$, reaching an end of its stroke when its centre line is tangent to the crank-pin circle; or when the crank is at either E or F. It will be seen that the driver passes over the arc FBE for the forward stroke, and through the arc EAF for the return stroke. The former arc is greater than the latter; hence the times of the strokes are in the ratio of these arcs, if the driver rotates uniformly.

The normal component only (v_2) of the crank-pin velocity (v_1) transmits motion to the follower; and $v_2 = v_1 \cos \phi$, in which ϕ is the angle OPO'. If a semicircle be drawn on OO' as a diameter, cutting $O'P$ at n, $Pn = OP \cos \phi$; hence $v_1 : v_2 :: OP : Pn$; or if OP represents the velocity of the crank-pin, Pn represents the velocity of the driven point of the slotted arm to the same scale.

The upper end of the slotted arm drives the cutter-bar of the shaper as indicated, through a pin, C, which is between two parallel projections attached to the cutter-bar. The velocity of C is v_3, perpendicular to $O'C$, and $v_3 : v_2 :: O'C : O'P$. To find this velocity draw a line through $O'v_2$ extended till it cuts Cv_3 in v_3. The motion of the tool, v_4, is the horizontal component of v_3. It differs little from v_3; but can be easily found by the graphical construction shown.

The fundamental portion of this mechanism is a modified form of the one used in the Whitworth motion; the only difference being that O', in this case, lies *outside* of the crank-pin circle; while in the other case it lies *inside* this circle. This difference in the proportions causes the slotted bar to vibrate through a definite angle in one case while it rotates continuously in the other case. The methods of connection with the ram of the shaper are quite different in these two cases, as is also the means of changing the length of the stroke. In the second form this change is made by changing the length of the driving crank-arm, means being

provided for moving the pin nearer to, or farther from, its centre, O. The adjustment can usually be made without stopping the machine.

With the Whitworth device, the relative time of forward and return strokes is not varied by changing the length of stroke. With the second mechanism the ratio between the times of the forward and return strokes is greatest with long strokes. The angle through which the driver passes for the forward stroke is $180° + \theta$, where θ is the angle of vibration of the slotted bar; and during the return stroke the driver passes through $180° - \theta$. The sine of $\frac{1}{2}\theta = OP \div OO'$, and as OO' is a constant, θ varies with changes of OP.

To design this machine, decide upon the ratio of the times to be occupied in the forward and return strokes for some particular length of stroke. Draw the crank-circle for this particular stroke (Fig. 177) and divide it into the arcs FBE and EAF, having this ratio. Draw tangents to this circle at B and F, and their intersection locates O'.

The velocity diagram is readily constructed for both strokes by finding the velocity $= v_4$ for various positions of the ram, by the method given. This diagram should be drawn as an exercise.

The crank and connecting-rod when arranged so that the centre line passes outside of the crank-circle centre (as discussed in Art. 110), may be used for a quick return. Elliptical gears (see Art. 46) are also used for quick-return mechanisms.

114. Bell-cranks.—Fig. 178 shows the method of designing a bent lever, or bell-crank, to transmit motion from line OA to line OB, with linear velocity ratio $= m \div n$. Lay off $Oa = m$ on OB, and $Ob = n$ on OA; complete the parallelogram $Obqa$ by drawing aa and bb parallel to OA and OB, respectively, and intersecting at q. Through O and q draw a line. Any centre, as Q, on this line may be taken as the bell-crank centre. From Q, drop perpendiculars QP and Qp on OA and OB; these are centre lines of the

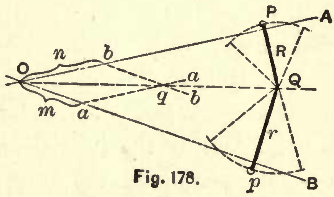


Fig. 178.

arms of the bell-crank. The angular motion of the arms should be the same on each side of QP and Qp.

It will be seen that any motion, either side of the central position, will shorten the effective arms. To reduce the deviation of the connected points to a minimum, the lengths R and r should be greater than QP and Qp, respectively, by one-half the versed sine of the angle described each side of the central position multiplied by the respective radii.

115. The Beam.—Fig. 179 indicates the beam of an engine; CC is the line of piston motion; $QF =$ distance of beam centre from this line, $= d$; $AB =$ stroke; $AE = \frac{1}{2}$ stroke $= s$; DF should $= EF$, to minimize the angularity of the connecting-rod. The beam length to fulfil this requirement is found thus: $QD = QA = d + EF$; but $\overline{QA}^2 = \overline{QE}^2 + \overline{AE}^2$. $\therefore (d + EF)^2 = (d - EF)^2 + s^2$. $\therefore d^2 + 2d \,.\, EF + \overline{EF}^2 = d^2 - 2d \,.\, EF + \overline{EF}^2 + s^2$. $\therefore 4d \,.\, EF = s^2$. $\therefore EF = DF = s^2 \div 4d$; $\therefore$ the length of the beam $= d + DF = d + \dfrac{s^2}{4d}$.

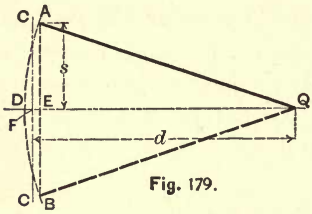


Fig. 179.

116. Rapid Change in Angular Velocity of the Follower.—A means of imparting rapidly changing angular velocity to a follower by the use of the linkwork is shown by Fig. 180. As C reaches the various positions in its path marked 0, 1, 2, 3, etc., c lies at $0'$, $1'$, $2'$, $3'$, etc., respectively, in its path. The principle of this arrangement is of frequent use. It is applied, as indicated in the figure, in the "wrist-plate" of the Corliss valve-gear, to give a rapid opening of the valves (and quick closing of the exhaust-valves) with a small motion when they are closed. In this application the rotating (vibrating) valve is attached to the arm $O' c$, and

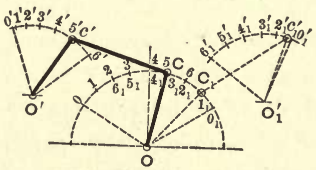


Fig. 180.

the mechanism is so proportioned that the required port opening is given quickly to a valve at one end of the cylinder, while the valve-arm at the other end moves but little during this period.

In general, the motion of the follower c is small compared to a given motion of the driver C as the angle $O' c C$ approaches a right angle and the angle OCc approaches 0 or 180°. On the other hand, the relative motion of c to C is great as the angle $O' c C$ approaches 0 or 180° and the angle OCc approaches 90°.

117. Straight-line Motion.—A large number of linkages have been devised to make a point move in a straight line independently of any planed guides.

The term *Parallel Motions* is usually given to such mechanisms, but straight-line motions is a more appropriate term. Fig. 181 shows what is known as Watt's parallel motion. R and r are arms centred at Q and q; Aa is a link connecting the free ends of R and r, and P is a point in Aa which traces an approximately straight line, within certain limits of the motion.

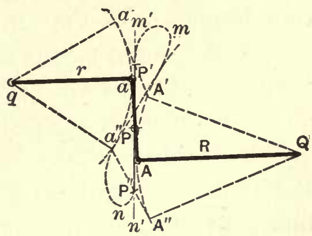


Fig. 181.

If R moves from its central position, A is drawn to the right, while the accompanying motion of r carries a to the left. The path of P is a function of both of these motions and the result is that P, if properly located in Aa, moves very nearly in a straight line, provided the angular motion of R and r does not exceed about 20°. The complete path of P is the "figure 8" shaped curve Pmn.

If $R = r$, $AP = aP$. In general, the segments AP and aP are inversely as the length of the adjacent arms.

Watt used this mechanism to guide the piston-rod in place of the slides now generally employed; but the principal application of "parallel motions" at present is on steam-engine indicator pencil motions. The Richards indicator, the earliest of the modern type, has the Watt mechanism.

The Tabor indicator has a motion in which a curved guide is

used ; it is, therefore, of a different type from the pure linkwork mechanisms usually classed as parallel motions. Fig. 182 indicates this pencil movement. It is desired that the pencil point P shall move in a right line, mm. It is evident that the curved guide nn can be given such a form that this will occur, and this curve can be found by moving P along mm, tracing the curve nn by the point p. Having found nn, a circular arc may be found which agrees closely with it, within the range of motion ; and if the centre of this arc be at d, a link dp can be substituted for the curved guide nn. An arrangement similar to this substitution is used on the Thompson indicator. If a moved in a straight line, instead of in the arc, yy ; if p were at the centre of Pa ; and if $dp = Pp = pa$, the mechanism would be the same as that shown in Fig. 172, and the result would be an exact straight-line motion ; requiring a straight guide, however, for the point a.

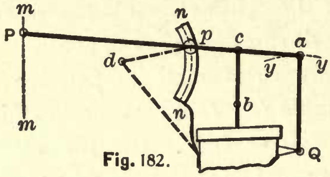


Fig. 182.

The Crosby indicator has a pencil mechanism similar to that of Fig. 183. If P be moved in a straight line mm, p (a point in the link bc) traces a curve ; the bridle link dp is one that will give an arc most nearly approaching this curve.

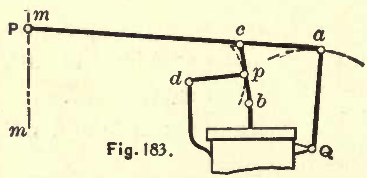


Fig. 183.

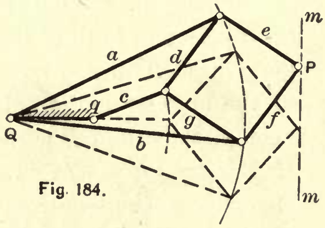


Fig. 184.

Peaucellier's straight-line motion is exact, and it is a pure linkwork. It is shown in Fig. 184. Two equal links a and b have a fixed centre, Q. The links d, e, f, g are equal ; and c, with a fixed centre at q, equals the distance Qq. P is constrained to move in the straight line mm.

118. Pantographs.—There are various linkages in which if one point is made to travel in any path, some other point will be constrained to describe a similar path, enlarged or reduced.

Fig. 185 shows one such arrangement in which $Pa = bQ$; and $aQ = Pb$. These links form a parallelogram which has a fixed centre at Q. A bar cd is attached to aQ and Pb parallel to Pa, and the point p, in cd, which lies on the line connecting P and Q, will move in a path similar to that traced by P. Suppose P to move to P', then p moves to p', and from similar triangles, $QP : Qp :: Qa : Qc$; also $QP' : Qp' :: Qa' : Qc'$; but $Qa = Qa'$, and $Qc = Qc'$ $\therefore$ $QP : Qp :: QP' : Qp'$, hence the distance of p from Q is proportional to the distance of P from Q. As p always lies in the line QP (because QaP and Qcp are similar triangles), the angular motion of p about Q is equal to the angular motion of P about Q. Any path of P is determined by its angular motion about Q and its radius vector to Q as a pole; as the angular motion of P and of p about Q are seen to be equal for any motion of either of these points, and as the radius vector of p bears a constant ratio to that of P, the path of p is similar to that of P.

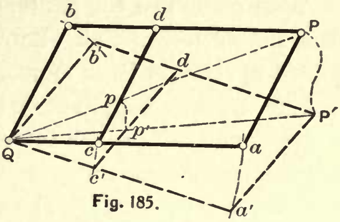


Fig. 185.

A form of pantograph, called the "lazy-tongs," is shown in Fig. 186. It is frequently used to reduce the piston motion of an

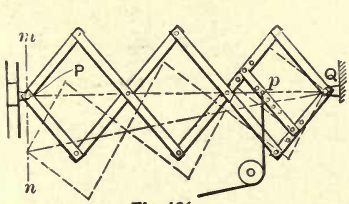


Fig. 186.

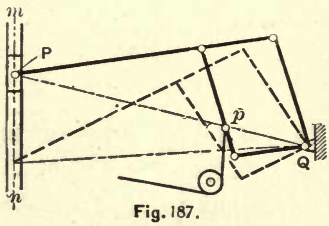


Fig. 187.

engine, in using the indicator. P is attached to the cross-head, and the indicator cord is attached at p. The practical objection to this contrivance is the great number of joints, and consequent liability to lost motion from wear.

Fig. 187 shows another pantograph for the same use. P is attached to the cross-head, and the cord is attached at p as before. With either of these arrangements the point p must lie in the line connecting P and Q, and the cord must be led off parallel to the cross-head motion.

Watt combined the pantograph with his straight-line motion so that the piston-rod, air-pump rod, and feed-pump rod were all guided in straight lines by means of one combination of links.

119. Hooke's Coupling, or the *universal joint,* is used for connecting two shafts which intersect. It is equivalent to what Reuleaux calls the four-link conic chain—that is, to a four-link chain in which the pivots are not parallel as in the ordinary case already treated, but their axes lie in radii of a sphere. Every point moves in the surface of a sphere, instead of in a plane. In its typical form (Fig. 188), each shaft has a forked end, and the two forks are united by an equal armed cross ab, cd, or its equivalent. The

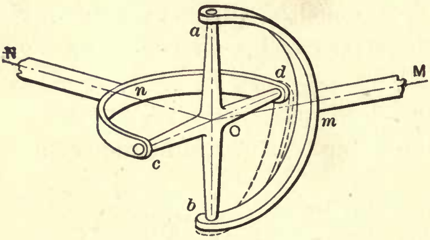


Fig. 188.

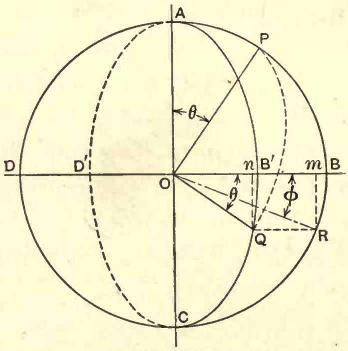


Fig 189.

shafts Mm and Nn and the arms of the cross (the axes of the pivots) intersect in a common point O. If only one half of each fork be considered, as mb of Mm and nd of Nn, and these are assumed to be connected by the spherical link bd equal to the fixed distance between the two adjacent points of the cross, a four-link conic chain is produced in which the axes of all the turning pairs intersect in O. With this arrangement the fork could be omitted, and

we would have the kinematic equivalent of the original mechanism.

The driver Mm and the follower Nn make complete revolutions in the same time; but the velocity ratio is not constant throughout the revolution.

If a plane of projection be taken perpendicular to the axis of Mm, the path of a and b will be the circle $ABCD$ in Fig. 189. If the angle between the shafts is β, the path of c and d will be a circle which is projected on the ellipse $AB'CD'$, in which $OB' = OD' = OB \cos\beta = OA \cos\beta$.

If one of the arms of the driver is at A an arm of the follower will be at B'; and if the driver-arm moves through the angle θ to P the following arm will move to Q; OQ will be perpendicular to OP, hence $B'OQ = \theta$. But $B'OQ$ is the projection of the *real* angle described by the follower. Qn is the real component of the follower's motion in the direction parallel to AC, which line is the intersection of the planes of the driver's and follower's paths. The true angle ϕ, described by the follower, while the driver describes the angle θ, can be found thus: draw QR parallel to OB so that $Rm = Qn$, then OR equals the radius of the follower, and $BOR = \phi =$ the true angle in plane $AB'CD'$ which is projected as $B'OQ = \theta$.

Now $\tan\phi = Rm \div Om$, and $\tan\theta = Qn \div On$; but $Qn = Rm$,

$$\therefore \frac{\tan\theta}{\tan\phi} = \frac{Om}{On} = \frac{OB}{OB'} = \frac{1}{\cos\beta},$$

$$\therefore \tan\phi = \cos\beta \tan\theta. \quad \dots\dots\dots \quad (1)$$

The angular velocity ratio of follower to driver is therefore found as follows by differentiation of Eq. (1), remembering that β is a constant in this equation:

$$\frac{\alpha'}{\alpha} = \frac{d\phi}{d\theta} = \frac{\cos\beta \sec^2\theta}{\sec^2\phi} = \frac{\cos\beta \sec^2\theta}{1 + \tan^2\phi}; \quad \dots \quad (2)$$

Eliminating ϕ by means of (1)

$$\frac{\alpha'}{\alpha}=\frac{\cos\beta\sec^2\theta}{1+\cos^2\beta\tan^2\theta}=\frac{\dfrac{\cos\beta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta\cos^2\beta}{\cos^2\theta}}$$

$$=\frac{\cos\beta}{\cos^2\theta+\sin^2\theta(1-\sin^2\beta)}=\frac{\cos\beta}{1-\sin^2\theta\sin^2\beta}. \quad . \quad (3)$$

By a similar process θ could be eliminated, giving

$$\frac{\alpha'}{\alpha}=\frac{1-\cos^2\phi\sin^2\beta}{\cos\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is seen from (3) that $\alpha' \div \alpha$ is a minimum when $\sin\theta=0$, or when $\theta=0$, π, etc., which corresponds to a value of $\phi=0$, π, etc. The same thing is seen from (4), which gives a minimum value of $\alpha' \div \alpha$ when $\cos\phi=1$, or $\phi=0$, π, etc. Also, $\alpha' \div \alpha$ is a maximum when $\sin\theta=1$, or $\cos\phi=0$, corresponding to $\theta=90°$, $\frac{1}{2}\pi$, etc.; $\phi=90°$, $\frac{1}{2}\pi$, etc.

To summarize the foregoing, the follower has a minimum angular velocity, if the driver has a uniform velocity, when the driving-arm is at A or C and the following arm is at B' or D'. The follower has a maximum angular velocity when the driving-arm is at B or D and the following arm is at A or C.

By using a double joint the variation of angular velocity between driver and follower can be entirely avoided. To do this an intermediate shaft is placed between the two main shafts, making the same angle, β, with each. The two forks of this intermediate shaft must be parallel. If the first shaft rotates uniformly, the angular velocity of the intermediate shaft will vary according to the law deduced above. This variation is exactly the same as if the *last* shaft rotated uniformly, driving the intermediate shaft; therefore, as uniform motion of either the first or the last shaft imparts the same variable motion to the intermediate shaft, uniform motion of *either* of these shafts will impart (through the intermediate shaft) uniform motion to the other. This is the combination used in the feed-rod of the Brown & Sharpe milling machines and elsewhere.

120. Ratchets.—The ratchet-wheel and pawl (Fig. 190) resemble both the direct-contact motions and linkwork. The driving-pawl CP acts by direct contact; but during driving the action is similar to that of a four-link chain, consisting of QC, qP, PC, and the fixed link Qq. Such mechanisms are sometimes termed *intermittent linkwork*

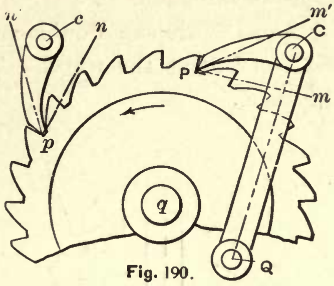


Fig. 190.

The two centres Q and q may coincide, the pawl-lever vibrating about the axis of the wheel. In this case there is no relative motion between the members during the forward (working) stroke.

The supplementary pawl, cp, has a fixed centre, c, and its object is to prevent the backward motion of the wheel when CP is not driving.

If pn is the common normal to the end of cp and the tooth with which it engages, there is no danger of the pawl becoming disengaged under the reaction of the tooth upon it; for the centres c and q are on the opposite sides of the line of action, and the tendency is for the wheel to run backward (right-handed rotation) and for the pawl to turn with a left-handed rotation, which only forces them together. If the direction of the common normal is pn', the centres both lie on the *same* side of the line of action, when the tendency is for both pawl and wheel to rotate in the right-handed direction, and the pawl would be forced out of contact, unless held by friction. In a similar way the normal Pm of the driving-pawl CP and the tooth on which it acts should pass between C and Q.

The pawl cp only prevents backward motion of the wheel after the wheel has moved back far enough to come in contact with the pawl. The amount of backward motion possible may vary from zero to the pitch of the teeth. This action could be limited by making the teeth small; but this would weaken the teeth, and the expedient is sometimes adopted of placing several pawls side by side on the pin, c, the pawls being of different lengths. With this arrangement the maximum backward motion may be reduced to the pitch divided by the number of pawls provided.

Sometimes, for feed-motions, etc., the pawl and wheel are made as shown in Fig. 191. This pawl can be reversed for driving in the opposite direction.

Fig. 192 shows a double-acting ratchet by which both strokes of the lever drive the wheel. The locking-pawl may be omitted in this case.

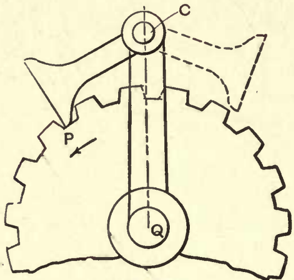
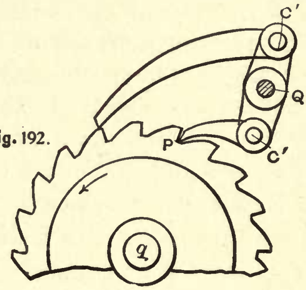


Fig. 191.

Fig. 192.


Frictional pawls (Fig. 193) are sometimes used, in which case the wheel is made without teeth. The pawl grips the wheel by friction during one stroke and releases it on the return stroke. These have the advantage of being noiseless, and the angular motion of the wheel for each stroke is not restricted to some multiple of the

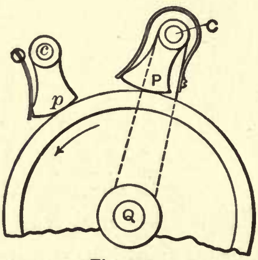


Fig. 193.

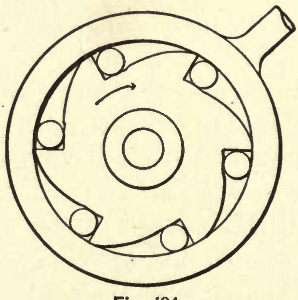


Fig. 194.

arc between two teeth, but the driving is not positive. Another frictional pawl with a fixed centre at *c* can be used to prevent "overhauling" of the wheel. The letters of Fig. 193 correspond with those of Fig. 190.

In the form of frictional ratchet shown by Fig. 194, the wheel is surrounded by a ring, which can be vibrated about the axis of the wheel. One of these members (either) has teeth of the form shown; and in the depressions formed by the teeth, rolls, or balls, are placed. Motion of the driver in one direction causes these rolls to bind the follower, while they release it on the return. Positive "silent" ratchets have been made with a special device for holding the pawl clear of the teeth on the return stroke.

The forms of ratchets shown by Figs. 190 to 195, and numerous modifications of them, are suitable for many cases requiring the conversion of a reciprocating action into an intermittent rotation. They are especially convenient in feed-mechanisms when the vibrations of the driver are not too rapid. At high speeds the shock between the pawl and tooth, as the driving begins, may be objectionable, and the inertia of the wheel is liable to make it move farther than desired, or to cause "overtravel." This last tendency prevents the employment of the ordinary ratchet when, as in revolution registers or continuous counters, a definite motion of the follower must be insured. A device for such purposes is shown by

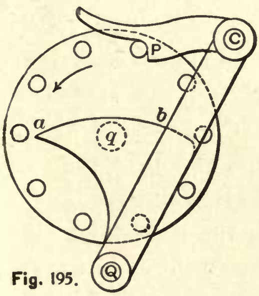


Fig. 195.

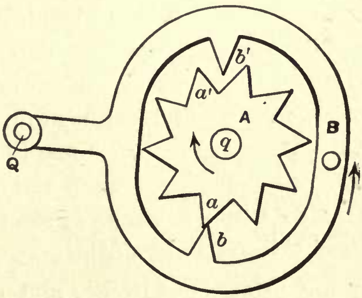


Fig. 196.

Fig. 195. The lever to which the pawl is attached has a tooth or beak so formed and placed that overtravel is impossible. When the pawl first acts on a pin, another pin passes close to the point of this beak; the beak then follows in behind this pin, crossing the path of pin-motions, and thus limiting the motion of the next pin. The outline ab should be a circular arc with Q as a centre, so that

the pin which it stops will rest against it during the return stroke of the driver. Another device much used for counters is shown by Fig. 196. The "star" wheel is driven through half of its pitch arc by the action of the projection b upon the tooth a during one stroke of the driver, and b' acts upon the opposite tooth a' during the return stroke, thus moving the wheel an equal distance in the same direction. It will be seen that the motion of the wheel for a double stroke of the driver is equal to the angle between two teeth, and if the wheel has ten teeth, it will make a complete rotation for ten double strokes of the driver.

121. Escapements.—The mechanism of Fig. 196 resembles the escapements used to control the motion of a train of clockwork, and it might, with slight modification, be used for such a purpose. If the wheel A is acted upon by a spring or weight which tends to rotate it continuously in the left-hand direction, this wheel would tend to produce reciprocation of the piece B. If B is a pendulum, it has a normal period of vibration corresponding to its length, and if the pendulum is so heavy that the rotative effort of A cannot alter this period, the pendulum in swinging will control the motion of the wheel. The tendency of the wheel to produce vibration of the pendulum may be made sufficient to overcome the frictional resistance which acts to stop the pendulum, and thus the amplitude of the vibrations is maintained. Other outlines of teeth for the wheel and pendulum are better, practically, and one common form is shown in Fig. 197. The teeth of the piece which vibrates with the pendulum are called *pallets*.

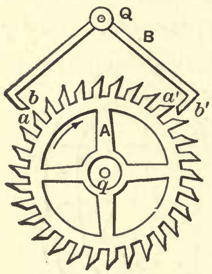


Fig. 197.

Many modifications of the escapement have been devised to meet special requirements. In watches and other portable timepieces a balance-wheel is used instead of the pendulum to regulate the period of the vibrating member, but all are similar in their general action.

CHAPTER VII.

WRAPPING-CONNECTORS. BELTS, ROPES, AND CHAINS.

122. Belts, Ropes, Chains, etc.—Flexible members are frequently used for transmission of motion between two pieces provided with properly formed surfaces upon which the connector wraps or unwraps as the action takes place. The connector may be a flat belt or band, a rope, or a chain composed of jointed members each one of which is itself rigid.

The great majority of the practical applications in which bands are used for transmitting motion are those in which the velocity ratio is constant. Figs. 198 and 199 show pairs of wheels of

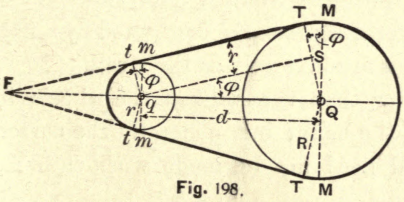


Fig. 198.

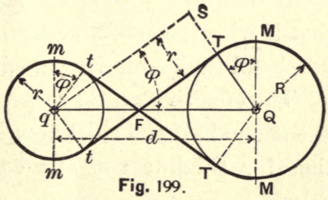


Fig. 199.

circular section connected by bands. These evidently fulfil the condition of constant velocity ratio, for the segments (QF and qF) into which the line of the band cuts the line of centres (or its extension) are constant; also, the perpendiculars (R and r) let fall from the fixed centres upon the line of the band are constant (see Art. 31). In case exact motion through only part of a revolution (or at most through a limited number of revolutions) is to be transmitted, the ends of the bands may be fastened to the wheels. The action, with this condition, is positive, provided the direction of the

motion is such that the band is always kept in tension. Thus in Figs. 55 and 56, the piece which rotates about O must be the driver, while the one rotating about O' is the follower, for transmission of motion in the directions indicated. The motions of two pieces connected in this way are necessarily of a reciprocating character, for when the band is all unwrapped from the follower the mechanism comes to rest, and any farther motion most be in the reverse direction. Such motion can only be secured when the former follower becomes the driver. An example similar to this case is seen in a hoisting-drum which pulls a car up an incline. While hoisting, the drum is the driver relative to the car; but in lowering, the action of gravity on the car causes it to turn the drum backward.

In most common applications of flexible connectors the ends of the band are joined together and not fastened to the wheels, and the motion is continuous; this is commonly called an endless band.

In these cases the motion is not positive, as the bands may slip (except when chains are used), but usually very exact motion is not essential where these devices are employed.

It follows from the demonstration of Art. 31, referred to above, that if wheels of circular transverse sections are connected by a flexible band their angular velocities are inversely as their radii.

The effective radii are greater than the radii of the wheels by about one half the thickness of the band; but generally the correction for thickness of the band need not be made with thin flat belts.

The *exact* effective diameter is the length of band that will just encircle the wheel divided by π. When a round cord or rope or a chain is used this affords a convenient way to get the effective or pitch diameter. Wheels for such ropes or cords have grooves cut in the rims to keep the band on the "sheave." For hemp or cotton rope transmissions the grooves are given such a form that the rope is wedged into them slightly, thus increasing the tractive force. With wire rope this wedging is inadmissible, as it would injure the rope, and the bottom of the groove has a somewhat larger radius than that of the rope. The bottom of the groove in wire-rope sheaves

is usually lined with rubber, leather, wood, or some such material, to increase the adhesion and save wear of the cable.

Fig. 200 shows the section of the rim of a sheave as commonly designed for hemp or other fibrous ropes. Fig. 201 shows a section of rim employed with wire ropes. If supporting sheaves or tighteners are required in a hemp-rope transmission the groove is made similar to that shown for wire rope but without the soft lining; for as these sheaves are not intended to transmit power, the increased adhesion due to the wedging of the rope is not required, and unnecessary wear of the rope is avoided by making the groove larger.

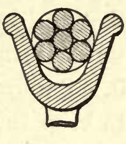


Fig. 200.

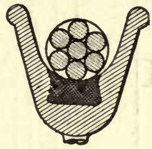


Fig. 201.

With chain-bands the wheels, called "sprocket" wheels, have projections fitting the links of the chain (more or less closely) to prevent slipping. With flat belts the pulleys have flat or nearly flat faces. The forms of sprocket-wheels and of the faces of pulleys for flat belts will be treated in later articles.

123. Shifting Belts.—If a pressure is brought to bear upon the advancing side of a belt (Fig. 202) the belt is deflected in the direction of this force. As the belt passes upon the pulley, each successive portion of it passes upon a part of the pulley farther from the side from which the shifting force acts, and the belt takes up a new position, as shown by the dotted lines. A pressure upon the receding side of the belt does not have this effect, unless the force is great enough to overcome the adhesion of the belt and pull it over bodily. It must be remembered, however, that the receding side of the belt relative to one pulley is the advancing side relative to the other pulley.

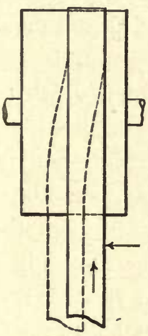


Fig. 202.

124. Crowning Pulleys.—If a flat belt is placed upon a cone (Fig. 203) the edge nearest the base of the cone is stretched more than the other parts, and the belt tends to take the position shown by the dotted line. The effect of this is to shift the belt towards

the base of the cone, as the advancing portion of the belt runs on nearer to the base.

If a similar cone is so placed that its base coincides with that of the first one, when the centre line of the belt has mounted to the common base it will remain in that position, as any displacement from such position would bring about the condition tending to return it. Pulleys are, therefore, usually made "crowning" to keep the belt on the centre. If the pulley is crowned about $\frac{1}{8}$ inch for each foot in width, the belt will ordinarily evince no tendency to run off, provided the axes of the connecting shafts are parallel. If the shafts are out of alignment, the belt tends to run toward the edges at which the belt is tightest, unless the shafts are very much "out."

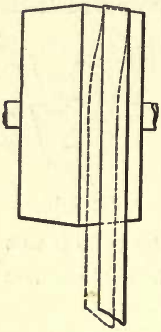


Fig. 203.

It is frequently desirable to stop the driven shaft without stopping the driver, and a common method of doing this is by means of "tight-and-loose" pulleys. Two pulleys are placed side by side on the driven shaft, one of which is fastened to the shaft, while the other is free to rotate relative to this shaft, but is prevented by collars from moving axially. The hub of the tight pulley usually serves as one of these collars, and the rims should not quite touch. A pulley is secured on the driving-shaft, opposite the tight-and-loose pulley, having a width (or face) equal to the combined width of both of the latter. A belt of about the width of either of the single pulleys connects one of them and the wide-faced driving pulley. When this belt is on the tight pulley, the follower is driven; but if it is shifted to the loose pulley the follower will stop, although the belt continues to run. The belt is easily shifted by applying a lateral pressure to the advancing edge, as explained in Art. 123. It is usual with tight-and-loose pulleys to make them both crowning, so that the belt will remain upon either when it is shifted; but to facilitate shifting the wide driving pulley is generally made with a straight face (cylindrical surface).

125. Length of Belt.—The length of belt is usually determined

by direct measurement if the pulleys are constructed and mounted, or by measuring a drawing if the work is not built and erected. This length may be calculated for either an open or a crossed belt (Figs. 198 and 199, respectively). This calculation is seldom of practical value simply for the determination of the length, but it plays an important part in the correct design of "stepped-cone pulleys," such as are used on the countershafts and spindles of lathes and other machines for securing changes of speed. The importance of this calculation will appear from the discussion of the next article.

The open band of Fig. 198 causes the follower to rotate in the same direction as the driver, while the crossed band (Fig. 199) gives the follower a rotation in an opposite direction. This will be seen to agree with the general statement of Art. 33; for with the open belt both fixed centres are on the same side of the line of action (the driving side of the belt); while, with the crossed belt these centres are on opposite sides of the line of action. Owing to the rubbing of the sides of the belt where they cross, the open band is used when it is feasible. The crossed band has the advantage of a larger arc of contact, which has an important effect on the adhesion, especially on the smaller pulley; but with wide, stiff belts, particularly when the distance between centres is small, the warping of the belt may largely destroy this advantage.

It is evident that the length of belt is different in the two cases, other conditions being the same. The following are the algebraic expressions for the length of belts:

The angle $MQT = mqt = SqQ = \phi$.

For crossed belts (Fig. 199),

$$\sin\phi = \frac{R+r}{d}, \text{ and } Tt = qS = \sqrt{d^2 - (R+r)^2}.$$

For open belts, $\sin\phi = \dfrac{R-r}{d}$, and $Tt = qS = \sqrt{d^2 - (R-r)^2}$.

The length of the crossed belt

$$= L = 2\sqrt{d^2 - (R+r)^2} + \pi R + 2R\sin^{-1}\frac{R+r}{d} + \pi r + 2r\sin^{-1}\frac{R+r}{d}$$

$$= 2\sqrt{d^2 - (R+r)^2} + (R+r)\left(\pi + 2\sin^{-1}\frac{R+r}{d}\right). \quad . \quad . \quad . \quad (1)$$

The length of the open belt

$$=L=2\sqrt{d^2-(R-r)^2}+\pi R+2R\sin^{-1}\frac{R-r}{d}+\pi r-2r\sin^{-1}\frac{R-r}{d}$$

$$=2\sqrt{d^2-(R-r)^2}+(R+r)\pi+(R-r)\left(2\sin^{-1}\frac{R-r}{d}\right). \quad . \quad . \quad . \quad (2)$$

It follows from (1) that a crossed belt which is of proper length for any pair of pulleys, R and r, will be of correct length for any other pair of pulleys, R' and r' (on the same shafts) if $R+r=R'+r'$, that is, if the sum of the radii is constant; for $(R+r)$ is the only variable quantity.

It will be seen, however, in (2) that if $R'+r'=R+r$; $R'-r'$ cannot equal $R-r$, unless $R'=R$ and $r'=r$.

An open belt of the correct length for two pulleys, R and r, on fixed shafts would not, therefore, be of exactly the right length for another pair of pulleys, R' and r', on these same shafts, if $R'+r'=R+r$, unless the two larger pulleys are equal, and the two smaller pulleys are also equal. Such a belt might be made to run if the distance between shafts were quite great and the change in sizes of pulleys were small; but it would not be equally tight on the different sets.

126. Stepped Cones.—It is often important to change the speed of a machine which is driven from a shaft having uniform speed. Cones, as shown in Fig. 204, might be placed upon the counter-shaft and on the spindle of the machine. If a crossed belt is used, it would be equally tight at all corresponding positions on these cones, but an open belt would not be; and in order to have it so, "swelled" cones, as shown (exaggerated) in Fig. 205, would be required. Such conical drums have the advantage of permitting every possible variation in speed within limits; but the belt tends to mount towards the large ends of both, which increases the strain upon the belts and the pressure upon the bearings.

The stepped cones, Fig. 206, are more compact than conical drums, and they avoid the objection just mentioned. It follows from the preceding discussion that for a crossed belt the sum of

the radii of any mating pair of steps should be a constant. But the sum of the radii of the intermediate pairs of steps should be greater than the sum for the outside steps when using an open belt. Rankine's Machinery and Millwork gives a method of determining the swell of the cones (Fig. 205) from which the radii of the intermediate steps of a stepped cone can be derived. A much

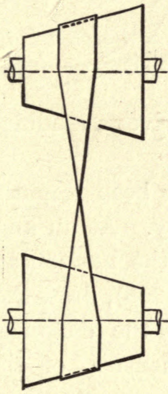


Fig. 204.

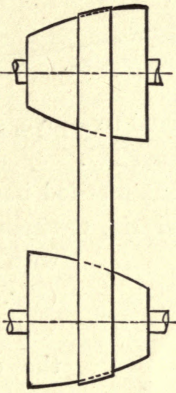


Fig. 205.

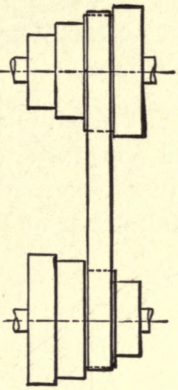


Fig. 206.

more convenient approximate graphical method is described by Mr. C. A. Smith, in the Trans. of the A. S. M. E., Vol. X, page 269.

Lay off Qq (Fig. 207) equal to the distance between shafts; draw the circles with radii R and r, equal to the radii of the known pulleys; at C, half way between q and Q, erect the perpendicular $CG = .314Qq$, and with G as a centre, draw the arc mm tangent to tT. The belt line of any other pair of steps should be tangent to mm. R' and r' are radii of two such steps; and the velocity ratio when using these steps will be $R' \div r' = FQ \div Fq$. Let $Qq = d$; let $Fq = x$; and call the desired ratio α.

Now $\dfrac{d + x}{x} = \alpha$. $\therefore\ x = \dfrac{d}{\alpha - 1}$. Lay off Fq equal to this value of x; draw FT' tangent to mm. Circles with Q and q as the respective centres, and tangent to FT', give the required wheels with

radii R' and r'. This method, as here outlined, only applies when the belt angle, ϕ of Fig. 207, is less than about 18°.

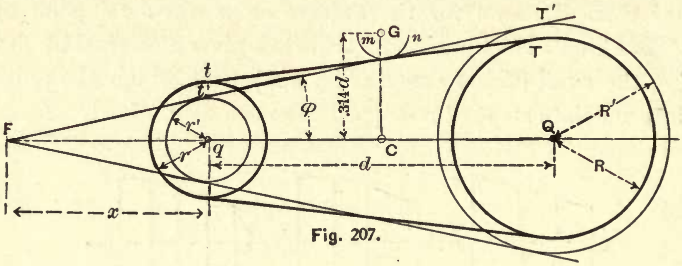


Fig. 207.

The original paper, referred to above, gives a modified method for use when ϕ is greater than 18°.

An even more convenient method has been recently devised by Dr. L. Burmester of Leipzig, Germany. At an angle of 45° with the horizontal line AB (Fig. 208), draw the line $AC = d =$ the distance between the centres of the shafts. Take $CD = \frac{d}{2}$, perpendicular to AC at C. With A as a centre draw a circular arc passing through D. On this arc locate E, so that the vertical distance between E and a point F, on AC, is $EF = R - r =$ the difference between the radii of the given pair of steps. Extend EF to G, making $FG = r$. Through G draw the horizontal OG, intersecting AC at O. Then $EG = R$, and $OG = r$. Draw OE. Let the angle $EOG = \theta$. Then $\tan \theta = EG \div OG = R \div r = n =$ the given velocity ratio. The radii R_1 and r_1 for any other velocity ratio n_1 are found as follows: Through O draw OE_1 at such an

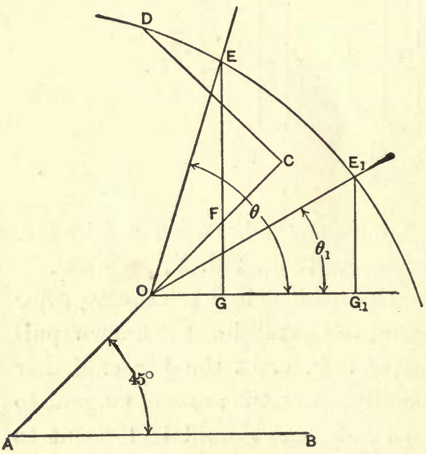


Fig. 208.

angle θ_1 with OG that $\tan\,\theta_1 = n_1$, intersecting the circular arc at E_1. Draw the vertical E_1G_1 intersecting OG produced at G_1. Then $E_1G_1 = R_1$ and $OG_1 = r_1$.

To secure satisfactory results the above construction should be accurately made to as large a scale as may be convenient. The values obtained should be checked by calculating the exact belt length for each pair of steps, using equation (3) of the preceding article.

127. Twisted Belts.—It is sometimes desired to connect two shafts which are not parallel by a belt. This can often be done by the use of a twisted belt. (See Fig. 209.) Suppose two pulleys in the plane of the paper (the lower one shown by the dotted circle) to be on parallel shafts, Q and q, Fig. 209.

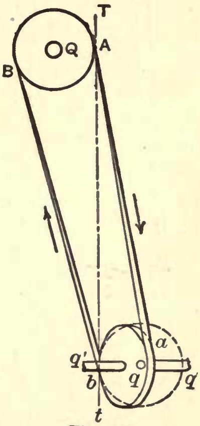


Fig. 209.

Draw Tt tangent to each pulley at the centre of its face, and on the side at which the belt *leaves* it. Then, if the lower pulley and its shaft be turned about Tt as an axis to the position shown by the full lines, the planes of the two pulleys will intersect in Tt. The line, Aa, in which the belt advances upon the lower pulley will lie in the plane of this pulley. The line, bB, in which the belt advances upon the upper pulley, will also lie in its plane. It has been shown (Art. 123) that the direction of the receding side of the belt does not affect the action; therefore this belt will remain upon the pulleys and continue to drive. If the motion of the pulleys be reversed, however, the belt will at once run off, because its advancing side does not lie in the plane of the pulley under this new condition. If the angle through which the lower shaft, $q'q'$, is turned is 90°, the term quarter-turn belt is applied.

128. Guide-pulleys.—The only condition necessary in order that a belt shall run on a pulley is that the centre line of its advancing side shall lie in the central plane of the pulley. By use of

guide-pulleys, or idlers, two shafts either intersecting at any angle or not in one plane can be connected by a belt. If desired, the belts can be made to run in *either* direction by so placing guide pulleys that *both* sides of the belt lie in the planes of the pulleys. Fig. 210 shows a few of the possible applications of guide-pulleys in connecting shafts which are not parallel. In the arrangement of Figs. 210 (*a*) and 210 (*c*) the belt may run in either direction; but

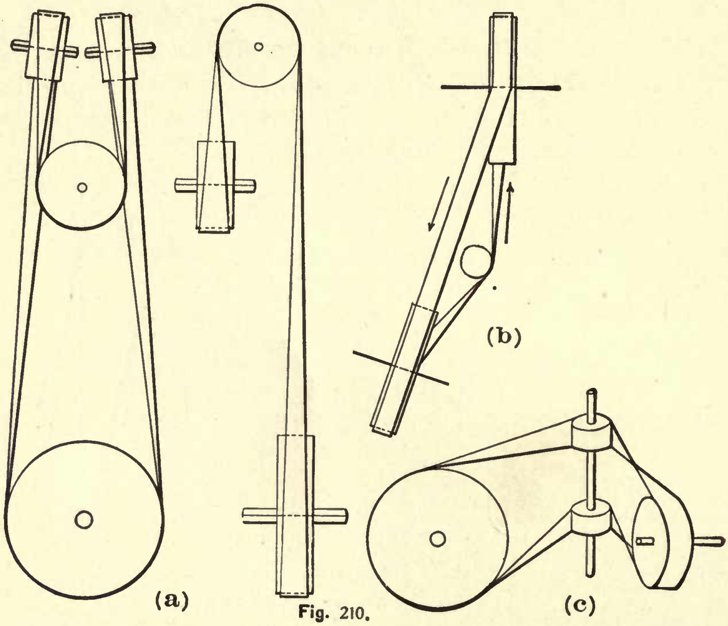


Fig. 210.

in Fig. 210 (*b*) the belt will only remain on the pulleys when it is run in the direction indicated by the arrows.

129. Belt-tighteners.—It is sometimes desirable to provide for variation in distance between shafts, to secure a greater arc of belt contact, to take up stretch of belt, or to avoid the use of clutches and tight-and-loose pulleys, by employing a belt-tightener. This is simply an idle pulley, mounted on a suitable frame in such a way that it can be moved by screws, levers, weights, or springs, to change or maintain the tension of the belt. The only condition necessarily complied with is that the centre line of the advancing

side of the belt shall lie in the central plane of the pulley to which it runs.

130. Sprocket-wheels for Chains.—One form of transmission-chain and sprocket-wheel is shown in Fig. 211. The true pitch line of the wheel is a closed equal-sided polygon, each side being equal to the length of a link from centre to centre of the pins. Or if a circle be drawn about Q passing through the centres of all the pins that lie on the wheel, the centre lines of the corresponding links form chords of this circle. As each link approaches or recedes from the wheel, one of its pin centres rotates, relative to the wheel, about the other pin centre, describing a circular arc relative to the wheel. Thus, Fig. 211, b describes the arc bp relative to the wheel as the link ab wraps upon the wheel. In order that the teeth of the wheel shall allow the links to drop smoothly into place, the actual tooth outline may be an arc parallel to pb, as shown by the arc mn. Adjacent sides of two teeth may be joined by an arc about p, the radius of which is equal to the radius of the pin, or bushing, which joins the links. By making the outer portions of the teeth lie somewhat inside the arcs mn, the pin does not rub upon the tooth as it approaches the wheel, but it will fall into place and reach a bearing at the end of its approaching action. The backs of the teeth are sometimes relieved more than the fronts or driving sides when the rotation is to be in one direction only. Since the true pitch line of the wheel is a polygon instead of a true circle, the velocity ratio is not exactly constant with sprocket-wheels. The irregularity is usually not important with wheels of a considerable number of teeth.

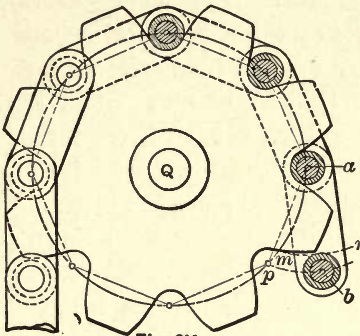


Fig. 211.

If two sprocket-wheels are connected by a chain, their angular velocity ratio is inversely as their numbers of teeth, as in toothed

gearing. This is a more convenient measure of the velocity ratio than the radii of the pitch circles, or the circles inscribing the pitch polygons.

Modifications of the construction shown in Fig. 211 permit the employment of chains with various forms of links, or of the special chains called "link-belts," etc.

A wheel frequently used in cranes for the common chain, with oval links of round iron, is shown by Fig. 212. Every other link lies on the wheel with its plane in the central plane of the wheel; while the intermediate links lie in planes normal to these. Pockets, as shown, prevent slipping, and the flanges at the sides strengthen the projecting teeth greatly, so that there is no difficulty in getting a wheel stronger than the chain itself.

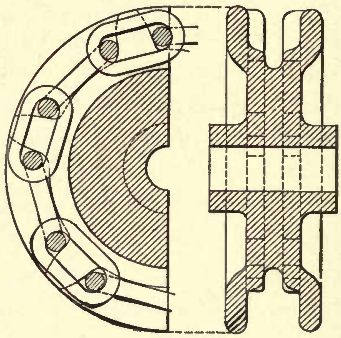


Fig. 212.

131. Wrapping-connectors with Varying Angular Velocity Ratio.—As already shown, flexible connectors can be used to transmit a variable angular velocity ratio, for instance, by using such forms as are shown in Figs. 54, 55, and 56. A somewhat different application is shown in Fig. 213. It has been employed in chronometers and watches to secure a more uniform driving action to the mechanism as the spring runs down. The spring is placed in the cylindrical piece, called the barrel, and as it uncoils the small chain is wound upon the barrel and unwound from the conical piece, called a "fusee." It will be seen that as the spring runs down the pull on the connector diminishes, but the "leverage" of the connector upon the follower is increased correspondingly, and, therefore, the driving effort transmitted to the mechanism may be kept quite uniform.

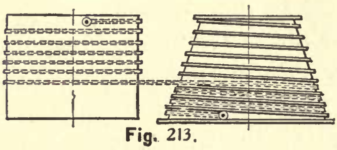


Fig. 213.

CHAPTER VIII.

TRAINS OF MECHANISM.

132. Substitution of a Train for a Simple Mechanism.—It is kinematically possible to transmit motion between two parallel shafts with any required angular velocity ratio by either a single pair of gears or of pulleys; but there are practical conditions which often make it desirable to effect the required transmission of motion by a series of mechanisms, or a compound mechanism, instead of by a single pair of gears, of pulleys, etc. Such an arrangement constitutes a *train of mechanism.* The train may contain pulleys with belts, ropes, chains, gears, screws, and linkwork, any or all; and it may be used to transmit motion between other members than parallel shafts. If two shafts are to be connected by gears, and the required velocity ratio is high, the difference in the size of the gears may be inconveniently great if a single pair is used. That is, the large wheel may occupy too much room, or be difficult to swing, or the small gear may have so few teeth that it would be objectionable. For example: suppose the velocity ratio is 25 to 1, and that strength requires wheels of 2 (diametral) pitch. Then if the pinion be given only 12 teeth, it will be 6 inches in diameter, and the large wheel will be $25 \times 6 = 150$ inches in diameter ($= 12\frac{1}{2}$ feet). Now, suppose that an intermediate shaft be introduced. This intermediate shaft can be connected to the slower of the original shafts by using a pair of gears which will cause it to rotate 5 times to 1 rotation of this primary shaft, and it can be connected to the faster of the original shafts by a pair of gears which will give it 1 rotation to 5 of the latter shaft; then as each

revolution of the first shaft corresponds to 5 revolutions of this intermediate shaft, and as each of its revolutions corresponds to 5 revolutions of the last main shaft, it is evident that the velocity ratio between the first and last shaft is 5×5 to 1, equal 25 to 1, as required.

The velocity ratio of first shaft to intermediate and of intermediate to last shaft are not necessarily equal. They may be anything whatever if the product of the separate angular velocity ratios equals the required ratio between the first and the last shaft. Furthermore, the three axes need not lie in one plane; that is, the centres need not be in one straight line. It is thus seen that the use of a train in place of a simple mechanism permits considerable flexibility in the arrangement; this will be clearly seen from an examination of various actual trains.

In a similar manner to that of the preceding illustration, an intermediate shaft may be used in a belt transmission when the velocity ratio is high. Such an arrangement is frequently seen when a slow-speed engine drives a dynamo. The engine is belted to a "jack-shaft," which in turn drives the dynamo. This may be desirable either to avoid an excessively large pulley or to avoid an extremely wide angle between the sides of the belt. The effect of a large belt angle is to reduce the arc of contact on the smaller pulley; this reduces the adhesion of the belt and increases liability of slip of the belt.

Other considerations than a high velocity ratio may make it desirable to substitute a train for a simple mechanism; for instance, to secure a required directional relation, for compactness, etc. A familiar example of such a train is seen in the back-gear mechanism (Fig. 217), as used on lathes and other machine tools.

A shaft which carries intermediate gears of a train may itself drive some member which requires a motion different from that of the last member. Thus, in clockwork, the gear on the shaft to which the minute-hand is fixed drives the hour-hand through a reducing pair of gears, and it may also drive a second hand at a higher rate.

133. Value of a Train.—Suppose four axes, I, II, III, and IV (Fig. 214) to be arranged as shown and connected by toothed gears of which the circles *a, b, c,* etc., are the pitch lines. The wheel *a* meshes with *b*; *c* meshes with *d,* and *e* meshes with *f.* Both of the wheels *b* and *c* are secured to the shaft II; hence they must rotate as one piece, having the same angular velocity at any instant. Likewise, *d* and *e* are both secured so shaft III, and they have the same angular velocity.

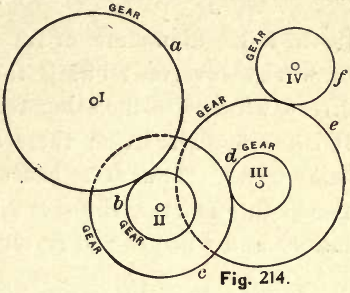


Fig. 214.

Let the angular velocities of the shafts I, II, III, and IV be represented by α_1, α_2, α_3, and α_4, respectively. Two gears which mesh together must have the same pitch; hence the numbers of teeth are proportional to the circumferences, to the diameters, or to the radii. But their angular velocities are inversely as the radii, and therefore inversely as the numbers of teeth on the wheel. It follows that if *a, b, c,* etc., are the numbers of teeth on the wheels designated by these letters, that.

$$\frac{\alpha_1}{\alpha_2} = \frac{b}{a}; \quad \frac{\alpha_2}{\alpha_3} = \frac{d}{c}; \quad \frac{\alpha_3}{\alpha_4} = \frac{f}{e};$$

$$\therefore \frac{\alpha_1}{\alpha_4} = \frac{\alpha_1}{\alpha_2} \times \frac{\alpha_2}{\alpha_3} \times \frac{\alpha_3}{\alpha_4} = \frac{b}{a} \times \frac{d}{c} \times \frac{f}{e} = \frac{b \,.\, d \,.\, f}{a \,.\, c \,.\, e} \quad . \quad . \quad . \qquad (1)$$

In this train *a* is the driver and *b* is the follower in the first pair; *c* is the driver and *d* the follower in the second pair; and *e* is the driver and *f* is the follower in the third pair. It will be seen from the above expression for $\alpha_1 \div \alpha_4$ that the angular velocity ratio of the first driving-shaft I to the last driven shaft IV equals the continued product of the numbers of teeth in the driven wheels divided by the continued product of the numbers of teeth in the driving wheels. The angular velocity ratio between two wheels i the direct ratio of the numbers of revolutions they make in a unit

of time, as a minute. In finding the value of a train, any of the factors, $\frac{\alpha_1}{\alpha_2}$, $\frac{\alpha_2}{\alpha_3}$, etc., may be expressed in terms of the numbers of teeth, radii, diameters, or revolutions per unit of time of the pair of wheels involved; but if the latter relation is used the ratio is direct, while with the other terms the inverse ratio is to be taken. It is not necessary that these different factors be all given in the same terms. Thus if a has 60 teeth and b has 16 teeth; c is 24 inches in diameter, and d is 8 inches in diameter; e makes 75 revolutions and f makes 250 revolutions per minute,

$$\frac{\alpha_1}{\alpha_4} = \frac{16}{60} \times \frac{8}{24} \times \frac{75}{250} = \frac{4}{15} \times \frac{1}{3} \times \frac{3}{10} = \frac{2}{75}.$$

For every revolution of I, IV makes $37\frac{1}{2}$ revolutions; hence if I makes 10 revolutions per minute, IV will make 375 revolutions per minute.

A train is shown by Fig. 215 in which the shaft I drives the shaft II through pulleys connected with a crossed belt; III is driven from II by an open belt; and IV is driven from III by gears. An expression similar to that given above can be used to

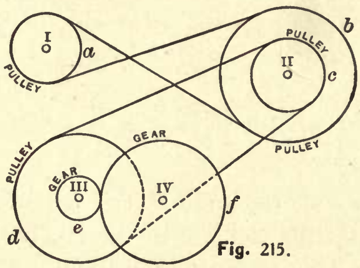


Fig. 215.

find the ratio of the angular velocities of I to IV. Thus suppose that the pulleys a, b, c, and d are, respectively, 8, 20, 10, and 24 inches in diameter; and that the gears e and f have 18 and 70 teeth respectively; then

$$\frac{\alpha_1}{\alpha_4} = \frac{20}{8} \times \frac{24}{10} \times \frac{70}{18} = \frac{70}{3}.$$

The shaft I makes 70 revolutions to 3 of the shaft IV (or $23\frac{1}{3}$ to 1). Or, if I makes 175 revolutions per minute, IV makes $\frac{175 \times 3}{70} = 7\frac{1}{2}$ revolutions per minute.

In general, if there are m shafts connected by gears or pulleys, the angular velocity ratio of the first shaft to the last is

$$\frac{\alpha_1}{\alpha_m} = \frac{\alpha_1}{\alpha_2} \times \frac{\alpha_2}{\alpha_3} \times \frac{\alpha_3}{\alpha_4} \ldots \frac{\alpha_{m-1}}{\alpha_m}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the numbers of revolutions per unit of time of the first and last shaft are N_1 and N_m, respectively, $N_1 : N_m :: \alpha_1 : \alpha_m$ (as the angular velocity of a member is proportional to its revolutions per unit of time); hence

$$N_m = N_1 \frac{\alpha_m}{\alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Belt connections are usually preferred when the speeds of the shafts are high, the distance between centres is great, and a moderate amount of slipping is not seriously objectionable. When the speed is slow, the distance between shafts is comparatively small, or when

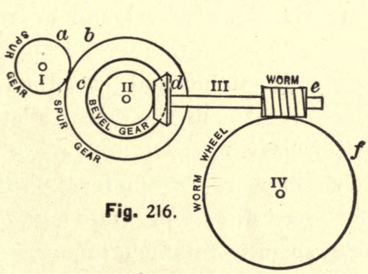


Fig. 216.

positive transmission is essential, gears are better. When this last condition is not a requisite, and the distance between shafts is too small to use belting advantageously, frictional gears are occasionally employed. When the distance between two shafts is very great, rope transmission (wire or hemp) may be used.

A train is shown in Fig. 216 in which the axes are not all

parallel. A pinion a on shaft I drives the spur-gear b on II; a pair of bevel-gears c and d connect II and III, and a worm e on III drives the worm-wheel f on IV. If the numbers of teeth on $a, b, c, d, e,$ and f are 15, 45, 25, 35, 1, and 50, respectively,

$$\frac{\alpha_1}{\alpha_4} = \frac{45}{15} \times \frac{35}{25} \times \frac{50}{1} = 210;$$

or the first shaft makes 210 revolutions to every revolution of the last shaft.

It will be seen that the expression for the value of a train, as deduced above, is general, and applies to all cases when the proper substitutions are made.

134. Directional Relation in a Train.—When two spur-gears mesh together they rotate in opposite direction ; hence, if the train is made up entirely of spur-gears the adjacent axes rotate in opposite directions, and the alternate axes (first, third, fifth, etc., or second, fourth, etc.) have rotations in the same direction. If such a train has an odd number of axes the first and last axes will rotate in the same direction; while if there is an even number of axes the first and last will rotate in opposite directions. Thus in the train of Fig. 214 the shafts I and IV rotate in opposite directions.

If one of the gears is an internal (or annular) gear the shaft to which it is attached rotates in the same direction as the pinion which meshes with this gear.

If an open belt connects the pulleys on two shafts these shafts rotate in the same direction, while a crossed belt connecting two shafts causes them to rotate in opposite directions. Thus in Fig. 215, I and II rotate in opposite directions ; II and III rotate in the same direction, and III and IV rotate in opposite directions. In this example there is an even number of shafts, but there is one open-belt connection ; hence, the rotations of the first and last shaft are in the same direction, as will appear from an inspection of the figure.

135. Back Gears.—The common screw-cutting lathe and many

other machine tools have a gear-train through which the stepped cone can be connected with the spindle. This is shown in Fig. 217. The cone A is driven by a belt from another cone on the counter-shaft. When the back gears are thrown out and the cone of the headstock is locked to the spindle C, these two members (the cone and spindle) move as one piece. If the cone has three steps the spindle can be given three different speeds from the uniformly revolving countershaft. By means of the back gears the number of speeds of the spindle is doubled without adding more steps to the cone. When the back gears are "in" the cone is not secured directly to the spindle, but is free to rotate upon it. A pinion, a, attached to the cone, engages with the first back gear, b, which is mounted on the shaft B. This shaft has another gear, c, secured to the opposite end; and c engages with the gear d, which is attached to the spindle. The angular velocity of the cone may be designated by α_1; that of the two back gears by α_2, and that of the spindle by α_3; then the angular velocity ratio of the cone to the spindle is $\frac{\alpha_1}{\alpha_3}=\frac{\alpha_1}{\alpha_2}\times\frac{\alpha_2}{\alpha_3}$; or $\frac{\alpha_1}{\alpha_3}=$ the product of the numbers of teeth on b and d divided by the product of the numbers of teeth on a and c.

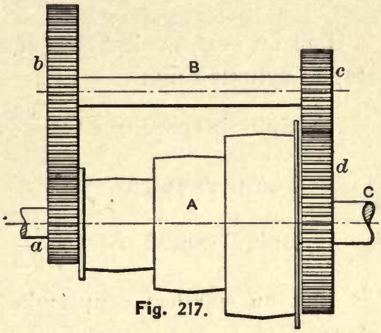


Fig. 217.

The cones on both the spindle and the countershaft are commonly equal with engine lathes; but on wood lathes (which do not use back gears) the countershaft cone is usually the larger, to secure the requisite high speed of the spindle from a moderate speed of countershaft.

A countershaft runs at 90 revolutions per minute, the four steps of the (equal) cones are 12″, 9½″, 7″, and 4½″ in diameter; the numbers of teeth on the gears a, b, c, and d are 28, 100, 24, and 88, respectively. The following speeds of the spindle may be obtained: Direct driving (back gears out):

Belt on largest step of countershaft cone and smallest step of spindle cone:

(1) Spindle speed $= 90 \times \frac{12}{4.5} = 240.$

Belt on next smaller step of countershaft cone and next larger step of spindle cone:

(2) Spindle speed $= 90 \times \frac{9.5}{7} = 122.14.$

Belt on next pair of steps:

(3) Spindle speed $= 90 \times \frac{7}{9.5} = 66.32.$

Belt on smallest countershaft step and largest spindle cone step:

(4) Spindle speed $= 90 \times \frac{4.5}{12} = 33.76.$

Driving through back gears: Value of back-gear train: $\frac{\alpha_3}{\alpha_1} = \frac{28 \times 24}{100 \times 88} = \frac{1}{13.1} = .076$ (nearly), giving four speeds with back gears which may be found by multiplying the four speeds as calculated above by $\frac{\alpha_3}{\alpha_1}$; or

(5) $= 240 \times .076 = 18.24.$
(6) $= 122.14 \times .076 = 9.4 -.$
(7) $= 66.32 \times .076 = 5.0 +.$
(8) $= 33.76 \times .076 = 2.56+.$

The student should take the necessary data from an actual lathe and compute the various spindle speeds.

136. The Idler.—It was shown in Art. 134 that one result of an intermediate shaft in a spur-gear train is to affect the direction of rotation between the first and third shafts. If these two shafts were connected directly by a pair of spur-gears they would rotate in opposite directions; but when connected through an intermediate shaft they rotate in the same direction.

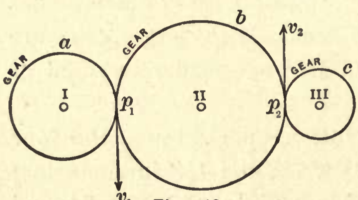


Fig. 218.

In Fig. 218 three shafts, I, II, and III, are shown connected "in series" by the gears *a*, *b*, and *c*. If these letters designate the numbers of teeth on the corresponding wheels :

$$\frac{\alpha_1}{\alpha_2}=\frac{b}{a};\quad \frac{\alpha_2}{\alpha_3}=\frac{c}{b};\text{ and }\frac{\alpha_1}{\alpha_3}=\frac{b}{a}\times\frac{c}{b}=\frac{c}{a},$$

or the intermediate wheel does not affect the ratio between the angular velocities of the first and third shafts, but it does cause them to rotate in the same direction.

Such an intermediate wheel in a train is called an *idler.* That the idler does not affect the ratio between the times of revolutions of *a* and *c* can be seen directly by inspection, for the linear velocity of a point in the pitch circle of *a* must equal that of a point in the pitch circle of *b*, and also points in the pitch circles of *b* and *c* must have the same linear velocities ; therefore, as all points in the pitch circle of *b* have the same velocity, the linear velocity of points in the pitch circles of *a* and *b* are the same, and the angular velocities of these two wheels are inversely as their radii, just as if they engaged directly.

Fig. 219 shows the "tumbling-gears" usually placed in the headstock of the screw-cutting lathe to enable the operator to easily change the direction of feed, or to cut either a right- or a left-handed screw. The gear *a* is connected to the lathe-spindle and *d* is on the stud through which the feed-rod or lead screw is driven. In the position shown, *a* drives *b*, *b* drives *c*, and *c* drives *d*. It will be seen that *a* and *d* rotate in opposite directions, and as *b* and *c* are both idlers, the action is equivalent to direct engagement of *a* and *d*. The gears *b* and *c* are carried on a support which can be swung about the centre of *d* by a suitable handle extending through the front of the headstock, and when this handle is dropped down, *c* can be meshed directly with *a*, *b* being thrown out of mesh with *a*.

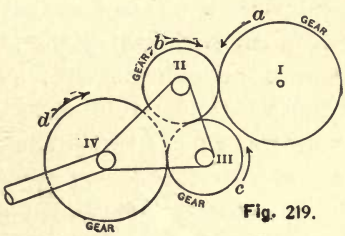


Fig. 219.

In this position b simply rotates, as it remains in mesh with c; but a drives c directly, and c drives d. There are but three axes in the train in this condition; hence a and d rotate in the same direction.

137. The Screw-cutting Train.—In the screw-cutting lathe a long screw, called the *lead screw*, or leading screw, is placed parallel to the bed, and the carriage which holds the lathe tool may be connected to this screw by a clamp-nut. When this nut is closed upon the screw the carriage will be fed along the bed as the screw is turned. If the screw has four threads to the inch ($\frac{1}{4}$-inch pitch), every turn of the screw will feed the tool $\frac{1}{4}$ inch parallel to the axis of the lathe. If the screw has the gear g (Fig. 220) mounted upon it at one end, the screw will make one revolution for each revolution of this gear. Now suppose the gear e to rotate with the lathe-spindle; then if e is equal to g, and is connected with it by the idler f, each revolution of the spindle compels the screw to make one revolution. If a cylindrical piece of stock is mounted in the lathe so that it rotates with the spindle, and a thread tool in the tool-post is fed (transversely) till it enters this cylindrical piece, it will be seen that the feed-mechanism will cause the tool to cut a thread on the stock which is a reproduction (as to pitch) of the leading screw; for the tool has a longitudinal motion of $\frac{1}{4}$ inch for each revolution of the work, and a proportional motion for any fraction of a revolution. The idler, f, is carried on a slotted piece which can be swung about the axis of the screw, VI, and the stud upon which f rotates can be set at different distances from VI, along the radial slot. The gear g could then be replaced by one of a different size, f could be moved along to engage with it, and by swinging the support of f it could also be made to engage with e, in which position it can be clamped. By this means the velocity ratio between the spindle and the gear can be varied.

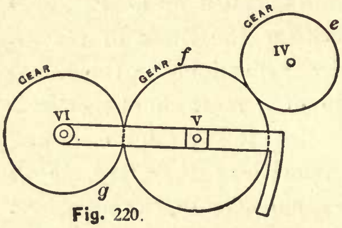


Fig. 220.

Suppose it is desired to cut a screw of 8 threads to the inch ($\frac{1}{8}''$ pitch). By placing a gear (g) twice as large as e on the screw,

each revolution of the spindle will cause g to make but half a revolution, and the tool will be fed only half the pitch of the leading screw along the stock during one complete revolution of the latter. To cut a screw of 6 threads per inch, g must be $1\frac{1}{2}$ ($\frac{3}{2}$) the size of e, then a revolution of the spindle and of the stock would occur for $\frac{2}{3}$ of a revolution of the screw; or the feed per revolution of the spindle would be $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$ of an inch. It will appear that screws of different pitches may be cut from a given lead screw, each of which is, in a sense, a reproduction, reduced or enlarged, of this screw.

The screw-cutting lathe is provided with a set of gears to be used as indicated above, for cutting all the whole number (even) threads throughout a rather wide range. Such a set is called a set of *change gears.*

A typical arrangement is a combination of the trains shown by Figs. 219 and 220. The gear a (Fig. 219) is on the spindle, and it drives d in either direction, through the tumblers, as explained in the preceding article. The stud (IV) to which d is attached passes through the end of the headstock and e (Fig. 220) is fastened upon its outer end. Then, by means of the change-gears any required thread within the range of the lathe can be cut, either right- or left-handed.

The gear on the outer end of the stud may be fixed, g only being changed; but provision is usually made for changing either e or g (or both). Sometimes a and d are not equal (d being usually twice as large as a in such cases); then the ratio between e and g must be taken accordingly. More often, however, a and d are equal.

A certain lathe of 16″ swing has a lead screw of 4 threads per inch, and change gears of the following numbers of teeth: 24, 30, 36, 42, 48, 48, 54, 60, 66, 69, 72, 78, 84. With the 24 gear on the stud it will cut: 5, 6, 7, 8, 9, 10, 11, $11\frac{1}{2}$, 12, 13, and 14 threads per inch, with the following gears, respectively, on the screw: 30, 36, 42, 48, 54, 60, 66, 69, 72, 78, 84.

The $11\frac{1}{2}$ thread corresponds to a standard pipe thread, and it is

consequently convenient to be able to cut this pitch in a lathe. To permit cutting this thread in the lathe, it is now not uncommon to provide a gear for it. It will be noticed that the above list of change-gears includes two 48-tooth gears. These are used for cutting a 4-thread screw, one of them being placed on the stud and the other on the screw. For cutting 2 (or 3) threads, one of the 48 tooth gears is put on the stud, and the 24 (or 36) gear must be used on the screw; as the screw must make 2 (or $1\frac{1}{3}$) revolutions, as the case may be, for each revolution of the spindle.

When the stud-gear e makes the same number of revolutions as the spindle, the following formula may be used for finding the change-gears, in which e equals the teeth in the stud-gear, g equals the teeth in the screw-gear, t equals the threads per inch of the lead screw, and n equals the threads per inch to be cut :—
$\frac{n}{t} = \frac{g}{e}$. If the stud-gear is fixed, $g = \frac{n}{t}e$. Any two gears of the set may be taken which have numbers of teeth in the ratio of n to t. If the stud-gear e does not make the same number of revolutions as the spindle, that is, if a and d of Fig. 219 are not equal, $\frac{n}{t} = \frac{d}{a} \times \frac{g}{e}$. The idlers, b, c, and f do not enter into the calculations, for they do not affect the velocity ratio.

The above screw-cutting train is given as an example of the ordinary arrangement; but among lathes of various makes there are many modifications in detail to be found. All ordinary screw-cutting lathes have a mechanism which is fundamentally that given above.

It will be noticed that in the series of gears given above there is a constant difference of 6 teeth between the successive gears (neglecting the gear for $11\frac{1}{2}$ threads and the extra 48-tooth gear). In any such system, for whole numbers of threads to the inch, this constant difference equals the number of teeth on the stud-gear divided by the threads per inch of the lead screw ($= e \div t$), when the spindle and stud have the same number of revolutions per unit of time. If the stud is geared to make only one revolution to two

revolutions of the spindle, the difference between successive wheels is half of that given by this rule.

The change-gears should always be constructed on the involute system, as this is the only system in which the centre distances can vary without affecting the constancy of the velocity ratio.

Many lathes are provided with a screw-cutting train which can be "compounded." In this arrangement the simple idler f (Fig. 220) is replaced by two gears of different diameters, secured together and rotating on the stud V as one piece. The gear e meshes with one of these intermediate gears, and the gear g (which must be correspondingly displaced laterally along its axis, VI) meshes with the other. This pair of intermediate gears (unlike the idler f) affects the velocity ratio between the spindle and the screw, because of the difference in the diameters of the two intermediate gears. The velocity ratio as found by the preceding method must be multiplied by the ratio of the two intermediate gears. This latter ratio is usually 2 to 1, or 1 to 2, depending upon whether the larger of the compounding-gears engages with e or with g.

138. Epicyclic Trains.—It was shown in Art. 39 that any member of a linkage could be considered as the fixed link, and apparently different mechanisms would be thus obtained. This is true of gear-trains as well as of linkages. If one of the gears of a gear-train is made the fixed member, instead of the bar supporting the gears, the mechanism is called an *epicyclic train,* because in its action one or more of the gears rotates on its axis at the same time that it revolves about the axis of the fixed gear, so that points in it describe epicycloidal curves.

Generally in these mechanisms the angular velocity ratio between the last rotating gear and the arm which carries it is required. In Fig. 221 let a and b be two gears mounted on an arm c, so that if c were fixed, a and b would form a simple gear-train. Now suppose that a is made fast to some fixed body, so that it really becomes the fixed member of the train. Then c can rotate around O, carrying b with it, b itself rotating relative to c around its

axis at O'. It is required to find the number of revolutions that b will make around its own axis, relative to the fixed member, for every revolution of c around O.

First, let a be disconnected from the fixed body, so that a, b, and c can make one revolution as one piece in the direction indicated around O. *Then b will make one revolution around O', solely because of its motion around O.* This can be seen by noting the

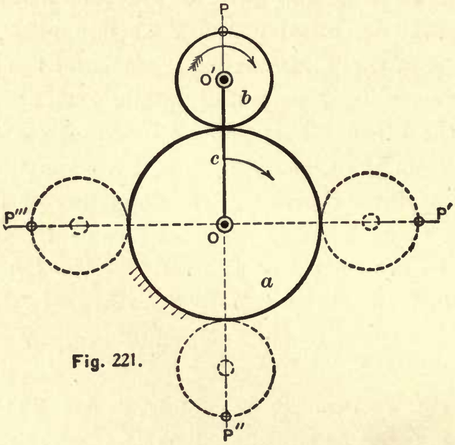


Fig. 221.

positions of any point, as P, relative to O' during different phases of the revolution, as shown. Now if c is held stationary and a is rotated backward one revolution, b will occupy the position it would have had if a had been held stationary all the time. Let r be the angular velocity ratio between b and a when c is held stationary. Then, when a is rotated backward one revolution, b must receive r turns forward, and the total number of revolutions which b will make for one revolution of c around $o = n = 1 + r$, and its *direction* of rotation will be the same as that of c. It is evident that c can be rotated in either direetion, and the result obtained above will still hold.

If we place an idler between a and b (Fig. 222), or if a is an annular gear (Fig. 223), the direction of motion of b is reversed so that it will make one turn in the direction of rotation of c and

minus r turns in the opposite direction, or $n = 1 - r$. The rotation of b (Fig. 222), relative to the fixed member, may or may not be in the same direction as that of c, depending on the value of r.

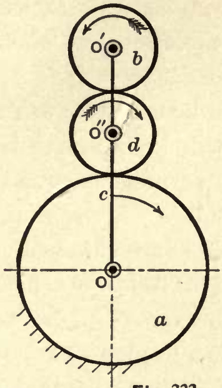


Fig. 222.

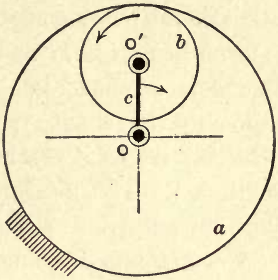


Fig. 223.

A special case is that when $r = 1$, whence $n = o$, and b does not rotate around O', relative to the fixed member, but has a simple motion of circular translation. The direction of rotation of b

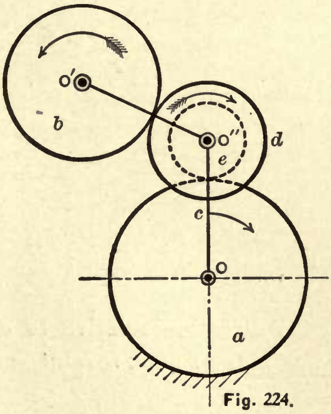


Fig. 224.

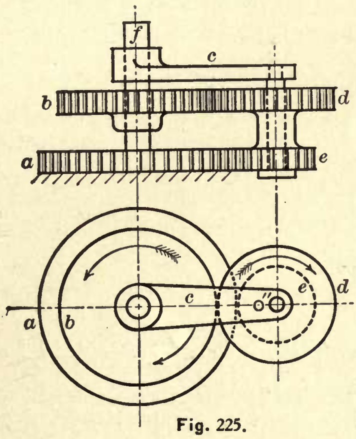


Fig. 225.

(Fig. 223) must always be opposite to that of c, since r can not be less than 1 when a is an annular gear.

In general, if the first and last gear of the train turn in opposite directions relative to the supporting bar, the last gear makes $1 + r$

revolutions in the same direction as the bar for each revolution of that member; when the first and last gears turn in the same direction relative to the supporting bar the last gear makes $1-r$ revolutions for each revolution of the bar. When $1-r$ is positive, the last gear and the bar turn in the same direction, but when $1-r$ is negative they turn in opposite directions.

It is evident that a compound train can be used between a and b, as shown in Fig. 224; and the results will be the same as with the arrangement shown in Fig. 222, since we are concerned only with the angular velocity ratio of b to a and the *direction* of rotation of b relative to a, regardless of how these are obtained.

Further, the axes need not lie in a straight line, but can occupy any position relative to each other as long as the gears mesh properly. A common arrangement is that shown in plan and elevation in Fig. 225, where the axis of b is made to coincide with that of the fixed gear a. The gears d and e are fast together, and are carried by c, which is free to rotate on the spindle f. It is, therefore, a compound chain, having three axes, and is called a *reverted train*.

In this form it is used extensively for obtaining great velocity ratios between the arm c and the last gear b.

For example :

Let a	have	99	teeth
" b	"	100	"
" d	"	101	"
" e	"	100	"

Then
$$r = \frac{99 \times 101}{100 \times 100} = \frac{9999}{10000};$$

and since a and b rotate in the same direction relative to c, $n = 1 - r = 1 - \frac{9999}{10000} = \frac{1}{10000}$ rev., or c must make 10,000 revolutions in order to make b rotate once.

The application of epicyclic gears to hoisting devices will be obvious from the above. They are also used for feed-mechanisms on large boring-bars, in machines for making wire ropes, etc., etc.

PROBLEMS AND EXERCISES.

NOTE.—A large number of exercises on Kinematics have been arranged by Mr. A. T. Bruegel, formerly of Sibley College, Cornell University, who has kindly consented to the use of some of them in the present work.

The original set contains three classes of exercises, intended:—to illustrate the principles treated; to drill the student on the application of these principles in the solution of definite problems, and to extend the range of the text. The exercises given below were selected mainly from those of the second class, and they include a few additional ones by the writer.

The references in brackets are to the articles in the text which relate most directly to the particular problem.

J. H. B.

1. [Art. 2.] A train has attained a speed of 112 miles per hour for a short distance. Express its velocity in *feet per minute*, in *feet per second*, and in *inches per second*.

2. [Art. 2.] The stroke of an engine is 18″, and the crank-pin makes 250 revs. per minute. Express the linear velocity, or rate of motion, of this pin in *feet per minute*; in *inches per second*; in *feet per second*.

3. [Art. 4.] The drivers of a locomotive are 5 feet in diameter, and the stroke of the piston is 24 inches. Calculate the *mean*, or *average*, piston speed (linear velocity) in *feet per minute* when the locomotive runs at the rate of 40 miles per hour.

4. [Art. 4.] An engine with a stroke of 5 feet makes 65 revs. per min. What is the mean piston speed?

5. [Arts. 4, 5, 6.] A train runs 110 miles in 2 hours and 40 minutes. Drivers, 64 inches in diameter. Stroke of piston, 22 inches. Required:

(*a*) Mean velocity of engine, in feet per minute, relative to the earth.
(*b*) Mean velocity of piston relative to engine-frame.
(*c*) Mean velocity of crank-pin relative to engine-frame.
(*d*) Mean velocity ratio between piston and crank-pin.

(*e*) Mean velocity of point in tread, relative to frame.

(*f*) Path of point in tread relative to frame.

(*g*) Path of point in tread relative to earth.

(*h*) Kind of motion of crank-pin and piston.

6. [Art. 14.] Represent, graphically, the mean velocity of the crank-pin of Prob. 5 (*c*). Use scale of 1000 feet per minute to the inch.

7. [Art. 14.] Represent, graphically, mean velocity of piston in Prob. 5 (*b*). Scale 700 feet per min. to the inch.

8. [Art. 17.] An engine with stroke of 18 inches makes 220 revolutions per minute. Find, graphically, the vertical and horizontal components of the crank-pin velocity when the crank makes angles of 30°, 120°, and 210° respectively, with its initial position on centre line of engine. Write the results in *feet per second* upon the lines which represent them.

9. [Art. 17.] A resultant pv (Fig. 18) equals 70 feet per second; the components pv_1 and pv_2 equal 64 feet and 48 feet per second, respectively. Find, graphically, the directions of the components. Two solutions are possible.

10. [Art. 17.] A velocity of 450 feet per minute is to be resolved into two components making angles with it, on opposite sides, of 30 and 60 degrees, respectively.

11. [Art. 17.] Three component motions in one plane have velocities of 60, 80, and 100 feet per minute, respectively; the first is vertically upward; the second makes an angle of 30 degrees to the right with it; and the third an angle of 45 degrees with the second, also to the right. Find the value of the resultant, graphically.

12. [Art. 17.] A point moving upward and to the right, at an angle of 60 degrees with the horizontal, has a velocity of 40 feet per minute.

(*a*) Resolve it into a vertical and a horizontal component.

(*b*) Resolve it into two components, one of which makes an angle of 45 degrees with the horizontal towards the right and has a velocity of 30 feet per minute.

(*c*) Resolve it into two components of 25 and 50 feet per minute, respectively. Graphical solutions required.

13. [Art. 17.] An engine of 24 inches stroke makes 160 revolutions per minute. The connecting-rod is four times the length of the crank. Find (graphically) the rate of motion of the cross-head when the crank is at 45 degrees and at 90 degrees with the centre line of engine.

14. [Art. 18.] A locomotive running at the rate of 35 miles per hour has 63-inch driving-wheels and 24-inch stroke. Find the linear and the angular velocity of the crank-pin *relative to the frame.* Give results in *feet per minute* and in *inches per second.*

15. [Art. 18.] An engine makes 600 *strokes* per minute. Fly-wheel is

on the crank-shaft. Find the angular velocity of the fly-wheel, the linear velocity of a point 3 feet from the centre of the shaft, and also of a point 4 inches from the centre. Express results in feet per minute.

When is the angular velocity of a point expressed by a number greater than that of the linear velocity?

16. [Art. 18.] A wheel 10 feet in diameter makes 100 revolutions per minute. What are the linear and the angular velocity of a point in the rim; of a point 6 inches from the axis; of a point 12 inches from the axis? Give all the results in feet per second.

17. [Art. 18.] (*a*) A body moves in a straight line with a linear velocity of 25 feet per second. What is its angular velocity?

(*b*) A governor-ball is 8 inches from the axis of rotation when revolving at the rate of 300 revolutions per minute. Express its linear and its angular velocity in units of *feet and minutes* and in *inches and seconds.*

18. [Art. 19.] Locate all the instant centres for the mechanism of Prob. 13, at the phases specified.

19. [Art. 20.] Same engine as Prob. 13, pressure on piston taken at 10,000 lbs. Draw the parallelograms of the forces acting upon the crank-pin and which *constrain* it to move in a prescribed path. Make a separate sketch for each of the following phases, the crank rotating clockwise $\theta =$ 45°, 150°, 210°, 300°, and the two positions at which the crank makes a right angle with the connecting-rod. [θ is the angle which the crank makes with the centre line of the engine.]

Also state whether the connecting-rod and crank are under tensile or compression stresses at each of the above positions.

20. [Art. 30.] An arm 12 inches long, rotating uniformly at 30 rev. per minute, drives an arm 30 inches long through an intermediate link 36 inches in length; distance between fixed centres 48 inches. Find, by method of instant centres, velocity of follower when driver-pin is on the line of and between the fixed centres, and 90, 180, and 270 degrees ahead of this position (4 phases). Also state the directional relation in each case.

Express velocities in feet per minute and tabulate results. Graphical solution.

21. [Art. 40.] Prove that, in the mechanism of Fig. 74, O_{ac} must lie in the intersection of the lines b and d (prolonged).

22. [Art. 41.] Draw velocity diagram for cross-head of an engine having stroke of 16″, and connecting-rod 40″ long. Engine makes 150 revs. per min. Prove for one ordinate. Also construct velocity diagram of cross-head with a connecting-rod 48″ long.

23. [Art. 41.] Fig. 77; $a = 6''$, $c = 14''$, $b = 18''$, $d = 20''$, and a makes

30 revs. per min. Construct velocity diagram for point O_{bc}; (*a*) upon its path as a base; (*b*) upon a rectilinear base. Prove for one ordinate.

24. [Art. 43.] Draw the pair of rolling centrodes for the relative motion of the cross-head and crank (Fig.70). Also draw the pair of centrodes for the relative motion of the connecting-rod and frame.

25. [Art. 46.] Two shafts are 6 inches apart; driver makes 50 rev. per min. Construct a pair of rolling ellipses for connecting the shafts, such that follower shall have a maximum rate of 75 rev. per min. What is the minimum rate of follower? Give major and minor axes of ellipses. (Draw pitch lines one-half size.)

26. [Art. 46.] Distance between fixed centres (opposite foci of ellipses) is 8″. Construct two rolling elliptical arcs, such that the velocity ratio will vary between the limits; 2: 3, and 4: 3, for an angular motion of the driver of 60 degrees.

27. [Art. 48.] Fig. 90. Take $O \ldots O' = 5''$, and $Op = 1\frac{1}{2}''$. Draw Ap perpendicular to Op, and construct the curve which will roll upon Ap; Ap and this curve to rotate about the fixed centres O and O', respectively.

28. [Art. 51.] Two parallel shafts, 24″ between centres, are to be connected by rolling cylinders. One shaft is required to make 350 revs. clockwise; while the other makes 500 revs. counter-clockwise. What are the proper diameters?

29. [Art. 51.] Same data as Prob. 29, except that the shafts are both to turn in the same direction. Required, the diameters.

30. [Art. 51.] Design rolling conical frusta to transmit motion between two shafts which intersect at an angle of 60 degrees. Driver to make 300 rev. to 400 rev. of follower. How may directional relation be changed without affecting the velocity ratio?

31. [Art. 55.] A pair of grooved friction-wheels have pitch diameters of 8 feet and 2 feet; working depth of groove equals $1\frac{1}{2}$ inches. The pinion makes 180 rev. per min. Find maximum sliding action, in feet and in inches per min.; assuming no slip at the pitch lines.

32. [Art. 62.] Epicycloidal gearing. Data:—pitch diameter of driver = 12″; of follower = 8″; 1 diametral pitch; addendum length of large wheel = 1″; of small wheel = $\frac{3}{4}''$; backlash = 0; bottom clearance = 0.1″; ratio of arc of approach to arc of recess = $\frac{4}{5}$; arc of action = circular pitch.

Required:—Diameters of describing circles; full construction of three teeth of each wheel; angles of maximum obliquity during both approach and recess.

33. [Art. 63.] Epicycloidal gearing. Data:—Pitch diameters 14″ and 10″; diameter of describing circles equal to radius of smaller wheel; back-

lash = clearance = $\frac{1}{16}''$; angles of approach and recess equal; $1\frac{1}{4}''$ circular pitch.

Required :—Least addenda which will insure contact between two pairs of teeth at all times ; angle of action in terms of the pitch. Test accuracy of the construction by rolling a tracing of one set of teeth upon the other.

34. [Art. 68, 69.] Annular involute gears. Construct several teeth of annular gear and pinion complying with the following conditions :

Diameters of pitch circles 12″ and 20″; 2 diametral pitch; clearance = $\frac{1}{16}''$; backlash = 0 ; addendum = $\frac{1}{2}''$; root = addendum + clearance ; profiles to be involutes line of action at 75° with line of centres), as far as possible; roots of pinion to be radial inside its base circle, outline of annular wheel teeth to be continued from the proper point by hypocloid of suitable form. Mark the point where this hypocloid joins the involute.

35. [Art. 77.] Approximate tooth outline. Data :—Circular pitch = 4″; number of teeth = 18 ; diameter of describing circle = radius of 12-tooth pinion; addendum = 33 pitch ; root = .37 pitch.

Required :—(*a*) Construction of tooth outline by the exact method ; (*b*) approximate (circular arc) outlines by the Willis, Grant, aud Unwin methods, for comparison. All of these outlines should pass through the same point on the pitch circle, and should be very carefully drawn with fine lines.

36. [Art. 77.] Draw outline of an involute tooth for a wheel 18″ diam. with 27 teeth. Compare this with Grant's approximation for involute teeth, by method similar to that outlined in Prob. 36.

37. [Art 79.] Design a mitre-gear (one of a pair of equal bevel-gears) with greatest pitch diameter = 10″ ; 20 teeth, epicycloidal outlines ; length of teeth along the elements equal to $2\frac{1}{2}$ times the circular pitch, and other dimensions with customary proportions. Thickness of rim equal to roots of teeth. Draw two views of one quarter of the wheel.

38. [Art. 90.] A No. 5 Brown & Sharpe cutter is used for involute wheels having from 21 to 25 teeth. Construct, accurately, the outline for one tooth of an involute gear of 1 diametral pitch, 21 teeth ; then with the same pitch and pitch points draw the outline for a wheel of 25 teeth. This comparison will show double the necessary maximum error in using one cutter through this range.

39. [Art. 90.] A No. 3 B. and S. cutter is used for wheels having 35 to 54 teeth. Make a construction (1 diametral pitch) for one tooth of each of these extreme sizes of wheels, and compare the difference with that found in Prob. 38.

40. [Art. 90.] Compare the maximum error in using an "M" cutter for

epicycloidal gears of 27 and 29 teeth, with the error in cutting 50 and 59 teeth by an " R " cutter. Use 3″ circular pitch.

41. [Art. 94.] Construct a cam on a base circle 3″ diam., to make one revolution per minute, and to impart to a roll 1″ diam., whose straight line of motion passes through the centre of the axis, a stroke of 2″. The roll is to rise uniformly during 25 seconds, remain at rest for 20 seconds, and descend during the remainder of the revolution with a uniformly accelerated motion (spaces passed over in equal times in the ratio of 1, 3, 5, 7, etc., as in falling bodies).

42. [Art. 94.] Draw a cam which by oscillating through an angle of 60° shall give a uniformly ascending and descending motion to a sliding-bar the line of motion of which passes 4″ to the right of the axis. Stroke of bar = 3″; base-circle = 10″. Cam acting on a roll 1½″ diam. at end of the bar.

43. [Art. 94.] The follower of a cam is a rocker, 22 inches long (roll 2″ diam. at free end), with fixed centre 4 inches above and 24 inches to the right of cam-shaft. Lowest position of follower is horizontal and cam rotates uniformly, moving follower through 30 degrees. During 90 degrees of rotation of cam follower describes angles in the ratio of 1, 3, 5, 3, 2, and 1, and then rests during the next 90 degrees of rotation of cam, and descends with uniform angular velocity during the remainder of rotation of cam.

44. [Art. 95.] A cam is to act upon a straight tangential follower, with The working face of the latter perpendicular to its line of motion. (See Fig. 142.) The follower is to be moved uniformly upward, a total distance of 1½″, while the cam rotates through 120°; then follower is to rest during an angular motion of the cam of 90°; and to descend with a uniformly accelerated motion during the completion of the rotation. Make base circle of cam = 4″.

45. [Art. 100.] Design a worm and wheel such that

$$\frac{\alpha}{\alpha'} = \frac{1}{50}; \quad \Delta = 5 \text{ ins.}; \ p = \tfrac{1}{2}''.$$

Required: T, t, D, d, ϕ.

Give the teeth the involute rack-and-pinion outline at middle section, and mark contact points of teeth.

46. [Art. 104.] If, in the four-link chain of Fig. 159, $a = 3''$; $b = 4''$; $d = 10''$; find the limits between which the length of c must lie in order to permit continuous rotation of a.

47. [Art. 104.] Taking same data as Prob. 46, is it possible to give c such a length that a drag-link mechanism results; that is, so that both a and b shall rotate continuously? Test this by finding the limiting values of c for the drag-link chain, with given values of a, b, and d.

48. [Art. 104.] If, in the drag-link chain of Fig. 160, $a = 7''$; $b = 6''$; $d = 3''$; find the limiting values of c which will permit continuous rotation of both a and b.

49. [Art. 106.] (See Fig. 164.) An engine with a stroke of 18″ has a connecting-rod 45″ long.

(*a*) Calculate the distance of the piston (or cross-head) from the end of stroke (*a–c*) when the crank angle (θ measured from A) is 60°.

(*b*) Calculate the distance from the other end of the stroke when $\theta = 120°$.

(*c*) Calculate the distance from cross-head to middle of the stroke (*q—m*), when $\theta = 90°$, or 270.

50. [Art. 106.] With data as in Prob. 50, except that connecting-rod is 54″ long, calculate (*a*), (*b*), (*c*).

51. [Art. 106.] Data as in Prob. 49. Calculate crank angles at which velocity of cross-head (piston) equals velocity of crank-pin. Also find ratio of piston velocity to crank-pin velocity when crank and connecting-rod form a right angle at C.

52. [Art. 110.] (Fig. 173.) The perpendicular distance from Q to the line of stroke, *h—g*, is 2″; radius of a crank = 3″; crank makes 20 rev. per minute; connecting-rod $C-c=9''$. Find length of stroke of *c*; and construct velocity diagram of *c* for forward and return strokes on *a—b* as a base.

53. [Art. 113.] (Fig. 175.) Design a Whitworth quick-return mechanism such that length of stroke *c* shall be 10″; ratio of times of forward and return strokes = 2 :1; rqadius of driving-crank (*OP*) = 4″; length of connecting-rod = 12″.

Construct velocity diagram of *c* for both strokes.

54. [Art. 115.] (Fig. 179.) The stroke of a beam-engine is 4 feet; distance from line of piston motion to beam centre (*d*) = 5 feet. Find proper length of beam for minimum obliquity of connecting-rod.

55. [Art. 126.] A countershaft runs at 100 rev. per minute. This countershaft is to drive a spindle through stepped cones and an open belt at 150, 100, or 75 rev. per minute. Largest step on countershaft = 14″ diam. Distance between centres = 7 feet. Find, graphically, the diameters of all the steps. Check the accuracy of the method by calculating the lengths of belts for each of the three pairs of steps.

56. [Art. 133.] (See Fig. 215.) The diameter of $a=24''$; $b=40''$; $c=36''$; $d=54''$; *e* has 15 teeth; and *f* has 48 teeth. Find velocity ratio and the directional relation between *a* and *f*.

57. [Art. 133.] (Fig. 214.) Data:—*a* has 60 teeth; *b* has 16 teeth; diam. of $c=24''$; diam. of $d=8''$; *e* makes 75 rev. per min. and *f* 250 rev. per min. How many rev. per min. does *a* make; and what is the directional relation between *a* and *f*?

58. [Art. 133.] (Fig. 216.) The number of teeth on *a*, *b*, *c*, *d*, *e*, and *f* are, respectively, 15, 45, 23, 35, 1, and 50. Determine velocity ratio between axes I and IV.

59. [Art. 135.] (Fig. 217.) The cone-pulley is driven by an equal cone on a countershaft which makes 90 rev. per min. The steps have diame-

ters of 12″, $9\frac{5}{8}$,″ and 7″. The gear a is keyed to the cone-pulley, and it has 28 teeth; gears b and c are fast to the shaft B, and have, respectively, 100 and 24 teeth; d is keyed to the spindle and has 88 teeth. Calculate the various possible speeds of the spindle.

60. [Art. 137.] The lathe has a lead screw with 4 threads per inch. The change-gears include wheels with the following numbers of teeth: 24, 30, 36, 42, 48, 48, 54, 60, 66, 69, 72, 78, 84. The " stud " makes the same number of revolutions as the spindle in a given time. With the 24-gear on the stud what gears should be used on the screw to cut 9, 10, 11, $11\frac{1}{2}$ and 12 threads, respectively? What arrangement would be used to cut 4 threads per inch? What for 2 threads?

61. [Art. 137.] Same data as Prob. 60. Arrange table showing what gears to use on the stud and screw to cut threads from 2 per inch up to 14 per inch.

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